Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Olusola Akinyele

Total contractive stability of differential systems over a finite time interval

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.5, p. 596–599.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_5_596_0>

Equazioni differenziali ordinarie. — *Total. contractive stability of differential systems over a finite time interval.* Nota di Olusola Akinyele, presentata (*) dal Socio G. Sansone.

Riassunto. — L'Autore dà una definizione di totale contrattiva stabilità in un intervallo di tempo finito e dà condizioni sufficienti perché un sistema differenziale abbia questa proprietà.

1. Introduction

In [1] Kayande gave necessary and sufficient conditions for the contractive stability of the following differential system, over a finite time interval:

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0, \cdots, t_0 \geq 0 \tag{1}$$

where f is defined and continuous on

$$\tilde{J} = [t_0, t_0 + T] \times R^n, \qquad T > 0, \qquad J = [t_0, t_0 + T)$$

and f satisfies a local Lipschitz condition on x. The notions of uniform stability and exponential contractive stability of the system (1) has also been defined and necessary and sufficient conditions obtained for the system (1) to possess these properties (cfr. 3]. However, the preservations of these various types of contractive stability under small perturbations have not found their way into the literature so far. The aim of this paper is to consider the preservation of contractive stability under small perturbations.

We shall consider along with the system (1) the perturbed system:

$$\frac{dx}{dt} = f(t, x) + R(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geq 0 \tag{2}$$

where f; $R \in C(\tilde{J} \times R^n, R^n)$, $C(\tilde{J} \times R^n, R^n)$ being the class of continuous functions from $\tilde{J} \times R^n$ into R^n. We shall define the notion of total contractive stability under permanent perturbations R, and prove a theorem which gives sufficient conditions for the system (1) to possess this property.

2. Preliminaries

We assume that $\|\cdot\|$ is a continuous non-negative function on R^n which need not be a norm and for η a real number, let

$$\overline{S(\eta)} = \{x \in R^n : \|x\| \leq \eta\}$$

(*) Nella seduta dell'8 maggio 1976.

be compact with respect to $\|\cdot\|$ and define

$$S(\eta) = \{x \in R^n : \|x\| < \eta\}.$$

We assume further that the solutions of (1) can be continued up to $t_0 + T$. With these notations we give the following definitions.

DEFINITION 2.1. Let α, β, η be given real numbers. Then the system (1) is said to be contractively stable with respect to $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$ $\beta < \alpha < \eta$ if $x_0 \in \overline{S(\alpha)}$ implies that

$$x(t, t_0, x_0) \in S(\eta) \quad \text{for all} \quad t \in J$$

and

$$x(t_0 + T, t_0, x_0) \in S(\beta),$$

where $x(t, t_0, x_0)$ is any solution of (1) through the point (t_0, x_0).

DEFINITION 2.2. Let α, β, η be given real numbers. Then the system (1) is totally contractively stable (stable under perturbations) with respect to $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$ $\beta < \alpha < \eta$, if for every solution $x(t, t_0, x_0)$ of the perturbed differential system (2),

$$x(t, t_0, x_0) \in S(\eta), \qquad \text{for all} \quad t \in J,$$

and

$$x(t_0 + T, t_0, x_0) \in S(\beta)$$

hold, provvided

$$x_0 \in \overline{S(\alpha)}, \|R(t, x)\| \leq \lambda(t), \lambda \in C(R^+, R^+), \qquad \text{for} \quad x \in \overline{S(\eta)}$$

and

$$\int_{t_0}^{t} \lambda(s)\, ds \to 0 \qquad \text{as} \quad y \to t_0 + T.$$

3. MAIN RESULT

We state and prove the main result of this paper.

THEOREM 3.1. *If the trivial solution* $x = 0$ *of* (1) *is contractively stable with respect to* $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$ $\beta < \alpha < \eta$, *then it is also totally contractively stable with respect to* $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$.

Proof. Suppose the trivial solution $x = 0$ of (1) is contractively stable, then by [1] there exists a function $V \in C(\tilde{J} \times R^n, R)$ which is locally Lipschitzian in x, such that

$$\text{(i)} \quad \limsup_{h \to 0+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h} \leq 0$$

$$\text{for} \quad (t,x) \in \tilde{J} \times \overline{S(\eta)};$$

$$\text{(ii)} \quad \max_{x \in \overline{S(\alpha)}} V(t_0, x) < \min_{\|x\|=\eta} V(t,x), \, t \in J;$$

$$\text{(iii)} \quad \max_{x \in \overline{S(\alpha)}} V(t_0, x) < \min_{x \in S(\eta) - S(\beta)} V(t_0 + T, x).$$

Let $x(t, t_0, x_0)$ be any solution of (2) such that $x_0 \in \overline{S(\alpha)}$, then it is routine to show that

$$D^+ V(t,x)_{(2)} \leq D^+ V(t,x)_{(1)} + M \| R(t,x) \| \qquad \text{for } (t,x) \in \tilde{J} \times \overline{S(\eta)},$$

where M is the Lipschitz constant for V, and $D^+ V(t,x)_{(1)}$ and $D^+ V(t,x)_{(2)}$ are the Dini derivatives of V with respect to the system (1) and (2) respectively.

Using condition (i), then

$$D^+ V(t,x)_{(2)} \leq M \| R(t,x) \| \leq M\lambda(t), \, (t,x) \in \tilde{J} \times \overline{S(\eta)}.$$

Suppose $x_0 \in \overline{S(\alpha)}$, does not imply

$$x(t, t_0, x_0) \in S(\eta) \qquad \text{for} \quad t \in J = [t_0, t_0 + T),$$

then there exist a first point $t_1 \in J$ such that

$$\| x(t_1, t_0, x_0) \| = \eta.$$

Integrating equation (3) from t_0, to t_1 we have

$$V(t_1, x) \leq V(t_0, x_0) + M \int_{t_0}^{t_1} \lambda(s) \, ds.$$

Hence,

$$\min_{\|x\|=\eta} V(t_1, x(t_1, t_0, x_0)) \leq V(t_1, x(t_1, t_0, x_0)) \leq V(t_0, x_0) + M \int_{t_0}^{t_1} \lambda(s) \, ds.$$

$$\min_{\|x\|=\eta} V(t_1, x(t_1, t_0, x_0)) \leq V(t_0, x_0) + M \int_{t_0}^{t} \lambda(s) \, ds, \, t_1 < t < t_0 + T.$$

Taking limits as $t \to t_0 + T$, we have

$$\min_{\|x\|=\eta} V(t_1, x(t_1, t_0, x_0)) \leq V(t_0, x_0) \leq \max_{x_0 \in \overline{S(\alpha)}} V(t_0, x_0)$$

which contradicts (ii). Hence

$$x(t, t_0, x_0) \in S(\eta) \quad \text{for} \quad t \in J. \tag{4}$$

Suppose $x(t_0 + T, t_0, x_0) \in S(\beta)$ is not satisfied, then $\|x(t_0 + T, t_0, x_0)\| \geq \beta$ and by (4),

$$x(t_0 + T, t_0, x_0) \in \overline{S(\eta)} - S(\beta).$$

Hence

$$\min_{x \in \overline{S(\eta)} - S(\beta)} V(t_0 + T, x(t_0 + T, t_0, x_0)) \leq V(t_0 + T, x(t_0 + T, t_0, x)) \geq$$

$$\leq V(t_0, x) + M \int_{t_0}^{t_0+T} \lambda(s)\, ds \leq V(t_0, x_0) +$$

$$+ M \left\{ \int_{t_0}^{t} \lambda(s)\, ds + \int_{t}^{t_0+T} \lambda(s)\, ds \right\}, \; t_0 < t < t_0 + T.$$

Taking limits as $t \to t_0 + T$ we have

$$\min_{x \in \overline{S(\eta)} - S(\beta)} V(t_0 + T, x(t_0 + T, t_0, x_0)) \leq V(t_0, x_0) \leq \max_{x_0 \in \overline{S(\alpha)}} V(t_0, x_0)$$

wich contradicts (iii) and so we have

$$x(t_0 + T, t_0, x_0) \in S(\beta),$$

which completes the proof.

Remark. The condition that $\beta < \alpha$ has not been used in the proof so that the result is also valid for $\alpha < \beta < \eta$. In this case we then have the total expansive stability of the system (1). For the definition of expansive stability see [2]. A corresponding definition of total expansive stability is obtained from Definition 2.2 with $\alpha < \beta < \eta$ and so we have the following result.

COROLLARY 3.2. *If the trivial solution $x = 0$ of the system* (1) *is expansively stable with respect to* $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$, *then it is also totally expansive stable under perturbations with respect to* $(\alpha, \beta, \eta, t_0, T, \|\cdot\|)$.

REFERENCES

[1] A. A. KAYANDE (1971) – *A Theorem on Contractive stability*, « Siam J. Appl. Maths. », *21* (4), 601–604.
[2] A. A. KAYANDE and J. S. WONG (1968) – *Finite time stability and comparison principles*, « Proc. Cambridge Philos. Soc. », *64*, 749–756.
[3] L. WEISS (1968) – *Converse theorems for finite time stability*, « Siam J. Appl. Maths. », *16*, 1319–1324.