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On a partition of an Euclidean half-space

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Geometria. — *On a partition of an Euclidean half-space* (*). Nota di ITALO CAPUZZO DOLCETTA e MASSIMO LORENZANI, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Con metodi geometrici si stabilisce l'esistenza di soluzioni per sistemi di complementarità degeneri.

INTRODUCTION

The partition theorem for Euclidean spaces due to H. Samelson, R. M. Thrall and O. Wesler, see [4] (1), is one of the most important results in the theory of complementarity since it characterizes the matrices $\mathscr{A}$ with positive principal minors among those for which the complementarity system

$$\text{(I)} \qquad \begin{cases} \boldsymbol{x} \geq 0 \\ \mathscr{A}\boldsymbol{x} + \boldsymbol{b} \geq 0 \\ x_i (\mathscr{A}\boldsymbol{x} + \boldsymbol{b})_i = 0 , \qquad i = 1 , \cdots , n, \end{cases}$$

has a unique solution for all $\boldsymbol{b} \in \mathbf{R}^n$ (see [3] for a wide bibliography on the subject).

However, in many interesting cases the system (I) is degenerate, that is $\mathscr{A}$ happens to be singular. This is the case, for example, when $\mathscr{A} = \mathscr{I} - \mathscr{P}$, where $\mathscr{I}$ is the identity matrix and $\mathscr{P}$ is stochastic. Such a situation occurs when an optimal stopping problem for a Markov chain is studied by means of complementarity system (see [1]).

Having in mind this situation the purpose of this Note is to obtain a partition theorem for an half-space of $\mathbf{R}^n$, and then determine a class of matrices for which this partition is possible, characterizing in this way the set of all $\boldsymbol{b} \in \mathbf{R}^n$ for which (I) is uniquely solvable.

1. Let $\mathscr{A}$ be a $n \times n$ matrix, $\mathscr{I}$ the $n \times n$ identity matrix and B_j a column vector belonging to the set $\{I_j , -A_j\}$, where I_j and $-A_j$ are the j^{th} column of $\mathscr{I}$ and $-\mathscr{A}$ respectively. Let us denote by pos $(B_1 , \cdots , B_n)$ the cone

$$\{\boldsymbol{v} \in \mathbf{R}^n \,/\, \boldsymbol{v} = \sum_{i=1}^{n} \lambda_i B_i , \lambda_i \geq 0\} ;$$

and by $K(\mathscr{A})$ the cone

$$\bigcup \text{pos} (B_1 , \cdots , B_n) ,$$

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(1) The numbers in [ ] send to the bibliography at the end of the paper.

where the union runs all over the 2^n possible choices of the n-tuple $(B_1, \cdots, B_n)$, Clearly $K(\mathscr{A})$ coincides with the set of all $\boldsymbol{b} \in \mathbf{R}^n$ for which (**I**) has a solution. Finally we denote by $K'(\mathscr{A})$ the cone

$$K(\mathscr{A}) - \{\text{pos}(-A_1, \cdots, -A_n)\} .$$

From now on we shall make the following assumption on the matrix $\mathscr{A}$ of the system (**I**):

(**H**)
i) *rank* $\mathscr{A} = n - 1$,
ii) *the hyperplane* $\pi = Im(\mathscr{A})$ *has equation* $\sum_{i=1}^{n} \alpha_i x_i = 0$, *with* $\alpha_i > 0$, $i = 1, \cdots, n$.

Observe that (**H**) implies $K(\mathscr{A}) \subseteq \overline{\pi}^+$, where $\overline{\pi}^+$ is the closure of the positive half-space determined by π.

In analogy with [4], we give the following

DEFINITION 1. *The* $2^n - 1$ *cones* pos $(B_1, \cdots, B_n)$, *with* $(B_1, \cdots, B_n) \neq (-A_1, \cdots, -A_n)$, *are a partition of the half space* $\overline{\pi}^+$ *if*

i) $K'(\mathscr{A}) = \overline{\pi}^+$.

ii) *The intersection of every pair of distinct cones is exactly the lower dimensional cone spanned by the common vectors.*

The following theorem is an adaptation of the mentioned result of [4] to the case of a matrix satisfying the assumption (**H**); the proof can be performed along the same line and is therefore omitted.

THEOREM 1. *The following conditions are equivalent*:

1) *The* $2^n - 1$ *cones* pos $(B_1, \cdots, B_n)$ *are a partition of* $\overline{\pi}^+$.

2) *For every choice of* B_j, $j = 1, \cdots, n-1$, *with* $B_j \neq -A_j$ *for some* j, *the hyperplane spanned by those vectors separates the two vectors of* $\mathscr{I}$ *and* $-\mathscr{A}$ *corresponding to the omitted index.*

3) *If* $\mathscr{B}$ *is the matrix whose columns are the vectors* $B_1, \cdots, B_n$, *then sign det* $\mathscr{B} = (-1)^s$, *where s is the number of the* $-A_j$*'s among* $B_1, \cdots, B_n$.

4) *The principal minors of* $\mathscr{A}$ *up to the order* $n - 1$ *included are positive.*

COROLLARY. *If one of the four equivalent conditions in Theorem 1 is satisfied, then the system* (**I**) *is uniquely solvable for all* $\boldsymbol{b} \in \pi^+$.

Proof. It is enough to observe that $\pi^+ = \overset{\circ}{\widehat{K'(\mathscr{A})}}$.

2. In this section we look for a condition which permits us to apply the previous Theorem 1. Precisely, we consider the following problem:

"*Given* n *vectors* $A_1, \cdots, A_n$ *on an hyperplane* $\pi \subset \mathbf{R}^n$ *of equation* $\sum_{i=1}^{n} \alpha_i x_i = 0$, $\alpha_i > 0$ *for every* i, *is the matrix* $\mathcal{A}$ *whose columns are* $A_1, \cdots, A_n$ (*not necessarily in this order*) *such that* $K'(\mathcal{A})$ *is a partition of* $\overline{\pi}^+$?"

To answer this question we introduce the cones K_i, $i = 1, \cdots, n$, defined by

$$K_i = \{ \boldsymbol{v} \in \pi \mid \boldsymbol{v} = \sum_{j \neq i} \lambda_j A_j, \lambda_j \geq 0 \}.$$

DEFINITION 2. *The* n *cones* K_i *are a partition of* π *if*

$$\pi = \bigcup_{i=1}^{n} K_i.$$

PROPOSITION 1. *The following conditions are equivalent:*

1) *The* n *cones* K_i *are a partition of* π.

2) *Each* $(n-1)$-*tuple of vectors* A_i *is linearly independent and the linear space spanned by any* $n-2$ *among the* A_i*'s separates the other two.*

3) $-A_i \in \overset{\circ}{K}_i$, $\quad i = 1, \cdots, n$.

4) $\sum_{i=1}^{n} \lambda_i A_i = 0$ *for some* $\lambda_i > 0$, $\quad i = 1, \cdots, n$.

Proof. The following implications are obvious: 1) $\Rightarrow$ 3) $\Longleftrightarrow$ 4). Let us show then that 3) $\Rightarrow$ 2) $\Rightarrow$ 1); observe that 3) implies that each $(n-1)$-tuple of A_i's is linearly independent. If one assumes that an $(n-2)$-tuple exists which does not separate the other two, say A_1 and A_2, it would follow that $-A_1$ is separated from A_2, that is A_1 cannot belong to K_1, which is a contradiction.

Secondly, if 2) holds, suppose that the cones K_i are not a partition of π. Then the boundary of $\pi - \bigcup_{i=1}^{n} K_i$ is determined by $(n-2)$-dimensional faces of certain cones. Consider one of these faces: this separates the remaining two vectors, say A_1 and A_2. Each interior point of this face will be also interior to the cones spanned by the face and A_1, A_2 respectively. Then, such a point would belong to $\pi - \bigcup_{i=1}^{n} K_i$, therefore $\pi = \bigcup_{i=1}^{n} K_i$.

Let $(C_1, \cdots, C_n)$ and $(A_1, \cdots, A_n)$ be two n-tuples of vectors of π both satisfying one of the equivalent conditions of Proposition 1; denote by H_i and K_i, $i = 1, \cdots, n$, respectively, the cones associated to the two n-tuples and let us fix an ordering for the vectors C_i.

DEFINITION 3. *The two n-tuples* $(C_1, \cdots, C_n)$, $(A_1, \cdots, A_n)$ *are said to be congruent if there exists a permutation of the* A_i*'s such that* $A_i \in \hat{\mathring{H}}_i$, $i = 1, \cdots, n$; *or, equivalently,* $C_i \in \hat{\mathring{K}}_i$, $i = 1, \cdots, n$.

PROPOSITION 2. *Let* π *be an hyperplane of* $\mathbf{R}^n$ *of equation* $\sum_{i=1}^{n} \alpha_i x_i = 0$, *with* $\alpha_i > 0$. *Then the vectors* C_i, $i = 1, \cdots, n$, *obtained by orthogonal projection on* π *of the vectors* I_i, *determine a partition of* π.

Proof. For every choice of $n - 2$ vectors among $C_1, \cdots, C_n$, the linear variety spanned by those is exactly the intersection of π with the hyperplane spanned by the $n - 2$ I_i's corresponding to the C_i's and the normal vector to π. This hyperplane separates the two remaining vectors, say I_1 and I_2, since by the assumption on π, its normal lies in $\widehat{\mathring{\mathbf{R}}^n_+}$. Consequently, their projections too are separated by the linear variety. Then, by ii) of Proposition 1 the thesis follows.

THEOREM 2. *Let* $(A_1, \cdots, A_n)$ *be a n-tuple of vectors in* π. *Assume that* $(A_1, \cdots, A_n)$ *satisfies one of the equivalent conditions of Proposition 1 and that* $(A_1, \cdots, A_n)$ *is congruent to the n-tuple* $(C_1, \cdots, C_n)$ *of the orthogonal projection on* π *of the vectors* $I_1, \cdots, I_n$. *Then,* $K'(\mathscr{A})$ *is a partition of* $\overline{\pi}^+$, *where* $\mathscr{A}$ *is the matrix whose columns are the* A_i*'s,* $i = 1, \cdots, n$, *in a suitable ordering.*

Proof. We shall make use of condition 2) of Theorem 1. To this purpose consider, without loss of generality, the $n - 1$ vectors $A_2, \cdots, A_k, I_{k+1}, \cdots, I_n$. Let π' be the hyperplane spanned by them and $\rho = \pi \cap \pi'$. By assumption, $C_1 \in \widehat{\mathring{K}_1}$ and C_1 will belong to one of the two half-hyperplanes determined by ρ. Of course, A_1 will be in the other half-hyperplane, since $(A_1, \cdots, A_n)$ is a partition of π. It necessarily follows then π' separates I_1 from A_1; the theorem is therefore proved.

3. Let us apply now the results of the previous sections to the complementary system (**I**). Let $\mathscr{P}$ be an irreducible non negative matrix, that is all its entries p_{ij} are non negative and does not exist a permutation matrix $\mathscr{M}$ such that

$$\mathscr{M}^{-1} \mathscr{P} \mathscr{M} = \begin{pmatrix} \mathscr{B}_1 & O \\ \mathscr{B} & \mathscr{B}_2 \end{pmatrix}$$

where $\mathscr{B}_i$, $i = 1, 2$, is a square matrix of order $1 \leq m_i < n$, (see [5]).

Let $\mathscr{D}$ denote the diagonal matrix with $d_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, \cdots, n$, at position (i, i). The irreducibility of a non negative matrix $\mathscr{P}$ has been characterized by I.M. Chakravarty (see [2]) in terms of $\mathscr{A} = \mathscr{D} - \mathscr{P}$; we recall his result in a slightly different but equivalent form, using our terminology:

A non negative matrix $\mathscr{P}$ is irreducible if and only if $\mathscr{A} = \mathscr{D} - \mathscr{P}$ satisfies the assumption (**H**).

THEOREM 3. *If $\mathscr{P}$ is a non negative irreducible matrix, then* $K'(\mathscr{D} - \mathscr{P})$ *is a partition of $\overline{\pi}^+$, where $\pi = \mathrm{Im}(\mathscr{D} - \mathscr{P})$.*

Proof. The thesis will follow from Theorem 2 once it is shown that the n-tuple of the column vectors $(-A_1, \cdots, -A_n)$ of $-\mathscr{A}$ (1) is congruent to the n-tuple $(C_1, \cdots, C_n)$ of Proposition 2, that is we have to show that $-A_i \in \overset{\circ}{\widehat{H}}_i$, $i = 1, \cdots, n$.

Consider the hyperplanes π_i, $i = 1, \cdots, n$, spanned by the vectors $I_1, \cdots, I_{i-1}, I_{i+1}, \cdots, I_n$, and their intersections π_i' with π which are linear varieties of dimension $n - 2$.

To be clear we look at the case where $i = n$. Then the linear variety π_i separates C_i from the remaining C_j's, $i = 1, \cdots, n-1$. In the semi-hyperplane determined by π_n' containing H_n, the linear varieties π_i' bound a closed convex cone contained in $\overset{\circ}{\widehat{H}}_n$. Moreover this cone is the intersection of π with the orthant of $\mathbf{R}^n$ whose elements have non negative components except for the n^{th} which is strictly negative.

Since the entries a_{in} of $\mathscr{A}$ are non positive for $i \neq n$ and strictly positive for $i = n$, because $\mathscr{P}$ is irreducible, it follows that $-A_n$ has non negative components except for the n^{th} which is strictly negative. But this means that $-A_n \in \overset{\circ}{\widehat{H}}_n$. Similar reasoning goes on for $i \neq n$ and the theorem is proved.

COROLLARY 1. *If $\mathscr{P}$ is a non negative irreducible matrix, then all the principal minors of $\mathscr{A} = \mathscr{D} - \mathscr{P}$ up to the order $n - 1$ included are positive.*

Proof. It follows from 4) of Theorem 1.

COROLLARY 2. *If $\mathscr{P}$ is a non negative irreducible matrix then the complementary system*

$$\begin{cases} \boldsymbol{x} \geq 0 \\ (\mathscr{D} - \mathscr{P})\boldsymbol{x} + \boldsymbol{b} \geq 0 \\ x_i((\mathscr{D} - \mathscr{P})\boldsymbol{x} + \boldsymbol{b})_i = 0, \qquad i = 1, \cdots, n, \end{cases}$$

has a solution for all $b \in \mathbf{R}^n$ such that

$$\sum_{i=1}^{n} \alpha_i b_i \geq 0,$$

(1) Observe that in this case the order of the A_i's is fixed.

where the α_i *are the positive coefficients of the equation of* $\pi = \mathrm{Im}\,(\mathcal{D} - \mathcal{P})$.

We observe that, if $\sum_{i=1}^{n} \alpha_i b_i > 0$ there exists a unique solution as follows from the Corollary of Theorem 1; if $\sum_{i=1}^{n} \alpha_i b_i = 0$, there is an infinite number of solutions, as it is easy to check.

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