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Restrictive Stability of the non-linear Abstract Cauchy Problem

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Equazioni differenziali ordinarie. — *Restrictive Stability of the non-linear Abstract Cauchy Problem.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà condizioni sufficienti per la stabilità restrittiva e la stabilità restrittiva asintotica del problema di Cauchy.

INTRODUCTION

In this paper we shall continue our study of the stability criteria of the abstract non-linear Cauchy Problem begun in [1] and [2]. We consider the Cauchy Problem:

$$\frac{du}{dt} = A(t)\,u + f(t\,,u) \cdots u(t_0) = u_0 \in D(A(t_0)) \tag{1}$$

where $f \in C(R^+ \times Y, Y)$ and Y is a Banach space.

In [3] the notion of restrictive stability which is of practical interest in certain situations was first introduced and sufficient conditions were given for such a concept to hold for the differential system

$$\frac{dx}{dt} = f(t\,,x) \quad , \quad x(t_0) = x_0 \tag{2}$$

where $f \in C(R^+ \times R^n, R^n)$. In this paper as a consequence of main global results obtained in [1] and [2] we give sufficient conditions for the restrictive stability and restrictive asymptotic stability of a self-invariant set $M \subset Y$, with respect to the abstract Cauchy Problem (1).

MAIN RESULTS

We shall assume that for each $t \in R^+$, $A(t)$ is a linear operator on Y with $D(A(t))$ depending on t. We also assume the existence of solutions $u(t\,,t_0\,,u_0)$ of the Cauchy Problem (1) for all $t \geq t_0$, a solution of (1) being a strongly differentiable function $u(t) \in D(A(t))$ which satisfies (1) for all $t \geq t_0$.

The following general results were obtained in [1] and [2] respectively:

LEMMA 2.1. *Assume that*

(i) $V \in C(R^+ \times Y, R)$ *and* $V(t\,,u)$ *is Lipschitzian in* u *and* $Y_0 \subset Y$ *is, open*;

(*) Nella seduta del 12 febbraio 1977.

(ii) *the set* $E \subset Y_0$ *is such that* $\bar{E} \subset Y_0$ *and* $H \subset \partial E$ *where* ∂E *is the boundary of* E;

(iii) $\alpha \in C(R^+, R)$ *is such that for* $(t, u) \in R^+ \times H$, $V(t, u) \geq \alpha(t)$;

(iv) $u_0 \in E$ *and* $V(t_0, u_0) < \alpha(t_0)$;

(v) *For each* $t \in R^+$, *all* $h > 0$, *the operator* $R[h; A(t)]$ *exists as a bounded operator defined on* Y *and for each* $u \in Y$ $\lim_{h \to 0} R[h; A(t)] u = u$;

(vi) $g \in C(R^+ \times R, R)$ *and for* $(t, u) \in R^+ \times E$,

$$D^+ V(t, u) \leq g(t, V(t, u));$$

(vii) *any solution* $y(t, t_0, y_0)$ *of the scalar differential equation*

$$\frac{dy}{dt} = g(t, y) \quad , \quad y(t_0) = y_0 \tag{3}$$

satisfies $y(t, t_0, y_0) < \alpha(t)$, $t \geq t_0$, *provided* $y_0 < \alpha(t_0)$;

Then there exists no $t^* > t_0$ *such that* $u(t, t_0, u_0) \in E$, $t \in (t_0, t^*)$ *and* $u(t^*, t_0, u_0) \in H$.

LEMMA 2.2. *Assume that*

(i) $V \in C(R^+ \times Y_0, R)$ *and* $V(t, u)$ *is locally Lipschitzian in* u;

(ii) $g \in C(R^+ \times R, R)$ *and for* $(t, u) \in R^+ \times Y_0$,

$$D^+ V(t, u) \leq g(t, V(t, u));$$

(iii) *there exists a set* $F \subset Y_0$ *such that* $u_0 \in F$ *implies* $u(t, t_0, u_0) \in Y_0$, $t \geq t_0$;

(iv) *the set* $H \subset Y_0$ *is such that* $\bar{H} \subset Y_0$ *and for* $(t, u) \in R^+ \times (Y_0 \sim H)$, $V(t, u) \geq \alpha(t)$ *where* $\alpha \in C(R^+, R)$;

(v) *for each* $t \in R^+$, *all* $h > 0$, $R[h; A(t)]$ *exists as a bounded operator on* Y *and for each* $u \in Y$

$$\lim_{h \to 0} R[h; A(t)] u = u;$$

(vi) *there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that for any solution* $y(t, t_0, y_0)$ *of the scalar differential equation* (3) *the relation* $y(t, t_0, y_0) < \alpha(t)$, $t \geq t_0 + T$ *holds.*

Then there exists a $T = T(t_0, u_0) > 0$ *such that* $u_0 \in F$ *implies* $u(t, t_0, u_0) \in H$ *for* $t \geq t_0 + T$.

DEFINITION 2.3. A subset $M \subset Y$ is said to be self-invariant with respect to the Cauchy-Problem (1) if $u_0 \in M$ implies $u(t) \in M$ for all $t \geq t_0$.

For any set $M \subset Y$, let $\rho > 0$ and define

$$S(M, \rho) = \{u \in Y : d(u, M) < \rho\};$$

where $d(u, M) = \inf_{v \in M} \|u - v\|_Y$.

DEFINITION 2.4. Let M and N be any two subsets of Y such that $M \subset N$. The set N is said to be

(i) restrictively stable on M with respect to the Cauchy Problem (1) if for any solution $u(t)$ of (1) and for every $t_0 \in R^+$, there exist positive numbers $\delta = \delta(t_0, \varepsilon)$ and $\beta = \beta(t_0, \varepsilon)$ such that $u_0 \in S(M, \delta)$ implies

$$u(t, t_0, u_0) \in S(N, \varepsilon) \cap S(M, \beta), t \geq t_0;$$

(ii) restrictively asymptotically stable on M if (i) holds and for every $t_0 \in R^+$, there exist positive numbers $\delta = \delta(t_0)$ and $\beta_0 = \beta(t_0)$ such that $\lim_{t \to \infty} u(t, t_0, u_0) = u^0 \in S(N, \varepsilon) \cap S(M, \beta_0)$ uniformly on $u_0 \in S(M, \delta)$.

Suppose M is a compact set in Y which is self-invariant with respect to the equation (1). We now give a set of sufficient conditions which ensure the restrictive stability of N on M with respect to the Cauchy Problem (1).

THEOREM 2.5. *Assume that*

(i) $V \in C(R^+ \times S(N, \rho), R^2)$, $V(t, u)$ *is locally Lipschitzian in u and for* $(t, u) \in R^+ \times S(N, \rho) \sim M$

$$D^+ V(t, u) \leq 0;$$

(ii) $b \in C(R^+, R^2)$ *and for* $(t, u) \in R^+ \times S(N, \rho) \sim M$,

$$V_1(t, u) \geq b_1(d(u, N)) \quad , \quad V_2(t, u) \geq b_2(d(u, M));$$

(iii) *whenever* $u \in M$, *for every* $(t, r) \in R^+ \times (0, \rho)$, $V_1(t, u) < b_1(r)$ *and for every* $(t, r) \in R^+ \times (0, \rho)$ *there exists* $\beta = \beta(t, r) > 0$ *such that* $u \in S(M, r)$ *implies* $V_2(t, u) < b_2(\beta)$;

(iv) *for each* $t \in R^+$ *and* $h > 0$ *(h small) the operator*

$$R[h; A(t)] = [I - hA(t)]^{-1}$$

exists as a bounded operator defined on Y *and for each* $u \in Y$

$$\lim_{h \to 0} R[h; A(t)] u = u;$$

Then the set N *is restrictively stable on* M *with respect to* (1).

Proof. Let $t_0 \in R^+$ and $0 < \varepsilon < \rho$. By hypothesis (iii) there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $u_0 \in S(M, \delta)$ implies $V_1(t_0, u_0) < b_1(\varepsilon)$ and also there exists $\beta = \beta(t_0, \delta)$ such that $V_2(t_0, u_0) < b_2(\beta)$. We now appeal to Lemma 2.1.

Set $Y_0 = S(N, \rho)$ and $E = [S(N, \varepsilon) \cap S(M, \beta)] \sim M$. Take $\partial H_1 = \partial E \cap \partial S(N, \varepsilon)$ and $\alpha(t) = b_1(\varepsilon)$, with $g(t, u) = 0$. Then all the conditions of Lemma 2.1 are satisfied and so the solutions $u(t, t_0, u_0) \ni u_0 \in S(M, \delta)$ are such that $u(t, t_0, u_0) \in H_1$ for all $t \geq t_0$.

Also setting $\partial H_2 = \partial E \cap \partial S(M, \beta)$ and $\alpha(t) = b_2(\beta)$ the conditions of Lemma 2.1 are again satisfied and so $u(t, t_0, u_0) \in H_2$ for all $t \geq t_0$ provided $u_0 \in S(M, \delta)$. Hence $u_0 \in S(M, \delta)$ implies $u(t, t_0, u_0) \in S(N, \varepsilon) \cap S(M, \beta)$ for all $t \geq t_0$, which completes the proof.

A set of conditions that guarantee the restrictive asymptotic stability of N on M is given in the following.

THEOREM 2.6. *Assume that*

(i) $V \in C(R^+ \times S(N, \rho), R)$ *and* $V(t, u)$ *is locally Lipschitzian in* u;

(ii) $g \in C(R^+ \times R, R)$ *and for* $(t, u) \in R^+ \times S(N, \rho) \sim M$

$$D^+ V(t, u) \leq g(t, V(t, u));$$

(iii) $b \in C(R^+ \times [0, \rho), R)$, $b(t, w)$ *is non-decreasing in* w *for each* $t \in R^+$ *and for* $(t, u) \in R^+ \times S(N, \rho) \sim M$,

$$V(t, u) \geq b(t, d(u, N));$$

(iv) *for each* $t \in R^+$ *and all* $h > 0$ *the operator* $R[h; A(t)]$ *exists as a bounded operator defined on* Y *and for each* $u \in Y$,

$$\lim_{h \to 0} R[h; A(t)] u = u;$$

(v) *there exists a* $T = T(t_0, y_0) > 0$ *such that any solution* $y(t, t_0, y_0)$ *of the scalar differential equation*

$$\frac{dy}{dt} = g(t, y) \quad , \quad y(t_0) = y_0$$

satisfies the inequality $y(t, t_0, y_0) < b(t, r)$, $t \geq t_0 + T$ *for every* $r \in (0, \rho)$.

Then the restrictive stability of N *on* M *implies the restrictive asymptotic stability of* N *on* M.

Proof. Assume that the set N is restrictively stable on M with respect to (1), then for any solution $u(t)$ of (1) and every $t_0 \in R^+$, there exist positive numbers $\delta = \delta(t_0, \varepsilon)$ and $\beta = \beta(t_0, \varepsilon)$ such that $u_0 \in S(M, \delta)$ implies

$$u(t, t_0, u_0) \in S(N, \varepsilon) \cap S(M, \beta), \qquad t \geq t_0.$$

Therefore $u(t, t_0, u_0) \in S(N, \varepsilon)$ and $S(M, \beta)$ all $t \geq t_0$. We now make an appeal to Lemma 2.2.

First set $F = S(M, \delta)$ in Lemma 2.2, then condition (iii) of the Lemma is verified. Let $t_0 \in R^+$ and $0 < \varepsilon^1 < \varepsilon$. Set $H_1 = S(N, \varepsilon^1)$ and $Y_0 = S(N, \varepsilon) \sim M$, then for $(t, u) \in Y_0 \sim H_1$ and the monotonicity of $b(t, r)$, $V(t, u) \geq b(t, \varepsilon^1)$. Now set $\alpha(t) = b(t, \varepsilon^1)$ the condition (iv) of the Lemma is satisfied. Conditions (i), (ii), (v) and (vi) of the Lemma are already part of the hypothesis, hence the Lemma is applicable to H_1.

Thus there exists $T_1 = T_1(t_0, u_0) > 0$ such that $u_0 \in F = S(M, \delta)$ implies $u(t, t_0, u_0) \in H_1 = S(N, \varepsilon^1)$, for $t \geq t_0 + T_1$.

Let $t_0 \in R^+$ and $0 < \beta_0 < \beta$. Set $H_2 = S(M, \beta_0)$ $Y_0 = S(M, \beta) \sim M$, then for $(t, u) \in Y_0 \sim H_2$ and the assumption on $b, V(t, u) \geq b(t, \beta_0)$. Setting $\alpha(t) = b(t, \beta_0)$ and proceeding as in the above arguments there exists $T_2 = T_2(t_0, u_0) > 0$ such that $u_0 \in F = S(M, \delta)$ implies $u(t, t_0, u_0) \in H_2 = S(M, \beta_0)$ for $\varepsilon \geq t_0 + T_2$.

Choose $T = \max\{T_1, T_2\}$, then $T = T(t_0, u_0) > 0$ and $u_0 \in S(M, \delta)$ implies $u(t, t_0, u_0) \in S(N, \varepsilon^1) \cap S(M, \beta_0)$ for all $t \geq t_0 + T$ which is the restrictive asymptotic stability of N on M. The proof is complete.

References

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[3] V. LAKHSMIKANTHAM and S. LEELA (1971) – *Global results and stability of motion*, « Proc. Camb. Phil. Soc. », *70*, 95–102.