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Intrinsic geometry of the quantum-mechanical "phase space", hamiltonian systems and Correspondence Principle

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Fisica matematica. — *Intrinsic geometry of the quantum-mechanical "phase space", hamiltonian systems and Correspondence Principle.* Nota di VITTORIO CANTONI, presentata (*) dal Socio C. CATTANEO.

RIASSUNTO. — Si mette in evidenza un'analogia strutturale fra lo spazio delle fasi della meccanica classica e lo spazio proiettivo hilbertiano della meccanica quantistica, analogia che consente di definire, per i valori medi delle osservabili quantistiche, parentesi di Poisson che risultano coerenti con le abituali relazioni di commutazione degli operatori associati. In base ad una formulazione precisa del principio di corrispondenza, data nel contesto dello schema di Mackey per la descrizione di un sistema fisico del tipo più generale, si è poi condotti ad un chiarimento del rapporto fra meccanica classica e meccanica quantistica che elimina, fra l'altro, il carattere puramente formale della corrispondenza fra parentesi di Poisson classiche e commutatori quantistici.

1. *Introduction.*

From the usual formulation of quantum mechanics, in which the pure states of a physical system are represented in Hilbert space by vectors determined *up to a complex factor*, it is possible, in principle, to derive an equivalent "projective formulation" in which the states are represented *one-to-one* on the projective space $\tilde{H}$ associated with the Hilbert space H of the theory. Though the cost for such an elimination of the redundancy is the loss of the linear structure, which would presumably make the projective formulation unhandy for actual calculations, the analysis of the intrinsic geometry of $\tilde{H}$, regarded as a real (finite or infinite-dimensional) manifold, sheds light on striking analogies with the phase-space Φ of a classical system, and suggests an extension of the scheme which gives rise to a common setting for the classical and the quantum theory.

In part I it is shown that, just like the classical phase-space, on account of the complex structure of H the quantum-mechanical "phase-space" $\tilde{H}$ has even dimension whenever it is finite-dimensional, and possesses an intrinsic skew-symmetric tensor field η (together with a riemannian metric which is degenerate in the classical case).

If A and B are generic observables, represented in H by the hermitian operators **A** and **B**, then their mean values $\hat{A}$ and $\hat{B}$, which are well-defined as functions on $\tilde{H}$ and directly determinable by experiment without reference to the underlying Hilbert space, have Poisson-brackets $[\hat{A}, \hat{B}]$ with respect to the skew-tensor η exactly equal of the mean value $\hat{C}$ of the observable associated with the hermitian operator $C = -2\,i\,(AB - BA)$.

(*) Nella seduta del 14 maggio 1977.

In part II the Correspondence Principle is precisely formulated in the context of Mackey's general scheme for the description of a physical system. The geometric relation between quantum systems and their classical analogues is analysed, and the relation between classical Poisson-brackets and quantum commutators is explained in this broader framework.

We do not discuss here the connection between the present approach to the Correspondence Principle and other related topics such as the explicit determination of the commutation relations for specific fields [6, 7] or the link between classical and quantum mechanics in terms of deformation theory [8, 9].

I. THE QUANTUM-MECHANICAL " PHASE-SPACE " AND POISSON-BRACKETS

2. *Geometric structure of the quantum-mechanical " phase-space "*.

Let $\omega \in \tilde{H}$ be an arbitrarily fixed state. Denote by ε_0 one of the unit representatives of ω in H, and consider an orthonormal basis $\{\varepsilon_0, \varepsilon_H\} \equiv \{\varepsilon_0, \varepsilon_1, \varepsilon_3, \cdots\}$ in H. Except for ε_0, the elements of the basis are labeled, for convenience, by *odd* positive integers only (1). If $P \in \tilde{H}$ is any of the states whose unit representatives α in H satisfy the condition

$$(1) \qquad \langle \alpha , \varepsilon_0 \rangle > 0 ,$$

the arbitrary phase factor in the definition of α can be uniquely fixed by the condition that the component of index 0 be positive, so that the state has a well-determined representation

$$(2) \qquad \alpha = x^0 \varepsilon_0 + \sum_H X^H \varepsilon_H \quad , \qquad (x^0 > 0) .$$

Denote by U_ω the region of $\tilde{H}$ constituted by all the states which satisfy condition (1): if $\tilde{H}$ is regarded as a real manifold, the real part $x^H = \operatorname{Re} X^H$ and the imaginary part $x^{H+1} = \operatorname{Im} X^H$ of the complex components of α constitute a system $\{x^h\} \equiv \{x^1, x^2, \cdots\}$ of local coordinates in $\tilde{H}$ with domain U_ω (1). x^0 is not an independent coordinate in U_ω, since

$$(3) \qquad x^0 = \left(1 - \sum_H \bar{X}^H X^H\right)^{1/2} .$$

If Q and R are states in U_ω, represented in H by the unit vectors $\beta \equiv y^0 \varepsilon_0 + \sum_H Y^H \varepsilon_H$ and $\gamma \equiv z^0 \varepsilon_0 + \sum_H Z^H \varepsilon_H$ respectively, $(y^0 > 0 , z^0 > 0)$, the connecting vectors **PQ** and **PR** have components $dy^0 \equiv y^0 - x^0$, $dY^H \equiv Y^H - X^H$ and $dz^0 \equiv z^0 - x^0$, $dZ^H \equiv Z^H - X^H$, and their scalar product in H is given by

$$(4) \qquad \langle \mathbf{PQ} , \mathbf{PR} \rangle = dy^0 dz^0 + \sum_H \overline{dY^H} dZ^H .$$

(1) H and all capital indices run over *odd* positive integers, from 1 to n if H is finite-dimensional, from 1 to ∞ otherwise. h and all lower-case indices run over *all* positive integers, from 1 to $n + 1$ or to ∞ according to the dimension of H.

If Q and R belong to a first-order neighbourhood of P, and the differentials dy^0, dz^0 are expressed in terms of the independent coordinates $\{x^h\}$ in U_ω, the expression (4) transforms into the bilinear form

$$\sum_{h,k}\left(\delta_{hk}+\frac{x^h x^k}{(x^0)^2}\right) dy^h\, dz^k + i\sum_{H}\left(dy^H\, dz^{H+1} - dy^{H+1}\, dz^H\right) \equiv \equiv \sum_{h,k}\left(g_{hk}+i\eta_{hk}\right) dy^h\, dz^k . \tag{5}$$

At the point ω, the elements of the matrices g_{hk} and η_{hk} are the components of a symmetric tensor $\boldsymbol{g}$ and a skew-symmetric tensor η, and it is easy to check that the matrices $g'_{h'k'}$ and $\eta'_{h'k'}$ which would have been obtained by performing the construction at the same point ω but in terms of a different basis $\{\varepsilon_{0'}, \varepsilon_{H'}\}$ of H, (so that $\varepsilon_{0'} = \exp(i\theta)\,\varepsilon_0$, $\varepsilon_{H'} = T^K_{H'}\,\varepsilon_K$ with the matrix $T^K_{H'}$ unitary), coincide with the transformed components $g_{h'k'}$ and $\eta_{h'k'}$ of the tensors $\boldsymbol{g}$ and η under the coordinate transformation $\{x^h\} \to \{x^{h'}\}$ in U_ω. Thus the riemannian metric $\boldsymbol{g}$ and the skew-field η are intrinsic geometric elements of $\tilde{H}$.

3. *Poisson-brackets of the mean values.*

Denote by A and B two observables, represented in H by the hermitian operators **A** and **B**, respectively. Set $\mathbf{C} = -2i(\mathbf{AB} - \mathbf{BA})$. If $\hat{A}$, $\hat{B}$ and $\hat{C}$ are the mean values of A, B and C, regarded as functions of the state on $\tilde{H}$, then

$$[\hat{A}, \hat{B}] = \hat{C}, \tag{6}$$

where the square bracket denotes the Poisson-bracket in $\tilde{H}$ with respect to the skew-field η.

In fact, at the generic point $\omega \in \tilde{H}$, in the local coordinates $\{x^h\}$, the skew-tensor η has canonical form, so that the associated contravariant skew-tensor $\tilde{\eta}$ (defined by the conditions $\sum_h \tilde{\eta}^{hk}\,\eta_{kl} = \delta^h_l$) has the form:

$$\tilde{\eta}^{hk} = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ 1 & 0 & -1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The Poisson bracket of any pair of functions f and h with respect to η is defined by the equation

$$[f, h] = \sum_{h,k} \tilde{\eta}^{h,k}\,\frac{\partial f}{\partial x^h}\,\frac{\partial h}{\partial x^k}.$$

In particular, if $\hat{A} = A_{00}(x^0)^2 + \Sigma(A_{0H}X^H + A_{H0}\bar{X}^H)\,x^0 + \Sigma A_{HK}\bar{X}^H X^K = A_{00}(x^0)^2 + x^0\,\Sigma A_{0H}(x^H + ix^{H+1}) + x^0\,\Sigma A_{H0}(x^H - ix^{H+1}) + \Sigma A_{HK}(x^H - ix^{H+1})(x^K + ix^{K+1})$, where A_{00}, A_{0H}, A_{H0} and A_{HK} are the matrix elements of the hermitian operator **A**, and $\hat{B}$, $\hat{C}$ are expressed in a similar way in terms of the corresponding matrix elements (according to the usual rule to compute the mean values), it is very easy to verify equation (6) at ω, and therefore on $\tilde{H}$ since ω was chosen arbitrarily.

The above result will allow a geometric interpretation of the Correspondence Principle and lead to a natural link between classical Poisson-brackets and quantum commutators [2].

II. THE CORRESPONDENCE PRINCIPLE IN MACKEY'S SCHEME

4. *Interpretation of Mackey's scheme.*

Consider a physical system $\Sigma = \{S, O, p\}$ described by the set S of its *states*, the set θ of its *observables*, and the function $p(A, \alpha, E)$ representing the probability that the measurement of the observable A on the state α give a result in the Borel set E of the real numbers R (Mackey, Ref. [2]). Physically we shall interpret the generic state α, considered at time t_0, as the result of well-specified modalities of "preparation" starting at time $t_0 - \tau_\alpha$, where τ_α is a non-negative number representing the duration of the preparation process. Similarly we shall associate the generic observable A, considered at time t_0, with a well-defined "measurement" process starting at time $t_0 + \tau_A$, where τ_A can now also be negative or zero and depends on the measurement process under consideration.

Two states, even if associated with distinct modalities of preparation, must be identified if, in correspondence with any fixed measurement, they give rise to identical statistical distributions of the results. An analogous identification applies to the observables. This justifies the following axioms (Mackey, Ref. [2] p. 62):

I_a) if $p(A, \alpha, E) = p(A, \alpha', E)$ for every A and E, then $\alpha = \alpha'$;

I_b) if $p(A, \alpha, E) = p(A', \alpha, E)$ for every α and E, then $A = A'$.

On account of the already mentioned interpretation of the functions p, it is also natural to require that the following conditions be satisfied:

II) $p(A, \alpha, \varphi) = 0$; $p(A, \alpha, R) = 1$; $p(A, \alpha, E_1 \cup E_2) = p(A, \alpha, E_1) + p(A, \alpha, E_2)$ whenewer $E_1 \cap E_2 = \varphi$.

It is understood that the preparation of any state and the measurement of any observable can be repeated as many times as one wishes, and the statistical distribution of the results, for a given observable A and a given

(2) A relation similar to (6) has been exhibited by Strocchi (reference [1]). However his quantum-mechanical "canonical coordinates" correspond to a "phase-space" which is just the underlying Hilbert space H, and is *not* in one-to-one correspondence with the physical states.

state α, does not depend on the particular instants $t_0', t_0'', t_0''', \cdots$ at which preparation and measurement are repeated.

If, without changing the preparation of the state α at time t_0, the measurement process associated with A is modified by triggering it at time $t_0 + t + \tau_A$ (with $t \geq 0$) rather than $t_0 + \tau_A$, a *new* observable A_t is defined, which coincides with A if $t = 0$. The statistical distribution of the results of A_t on the state α depends in general on t, but is still independent of the instants $t_0', t_0'', \cdots$ at which the processes of preparation and measurement are repeated.

As t varies, the observables A_t constitute a one-parameter family which will be called the *time evolution* of A.

For any fixed state α and any fixed observable A, to every t there corresponds a well-determined statistical distribution $p(A_t, \alpha, E)$, with an associated mean value $\hat{A}_t(\alpha)$. If, as we shall assume, the function $\hat{A}_t(\alpha)$ is differentiable with respect to t at the *initial time* $t = 0$, the function $\dot{\hat{A}} \equiv \left(\frac{\partial \hat{A}_t(\alpha)}{\partial t}\right)_{t=0}$ is well-defined on S.

In our considerations the question of mutual "compatibility" of distinct observables will never arise, since any single preparation process will always be thought as followed by *a single* measurement process, although, as already stressed, any given preparation-measurement pair can be repeated as many times as one wishes.

5. *Classical and quantum systems.*

We shall say that A is a *classical observable* if there exists a real function $A(\alpha)$, the *value of A* on the state α, such that $p(A, \alpha, A(\alpha)) \equiv 1$. Obviously, in this case, $\hat{A}(\alpha) = A(\alpha)$, and by means of the measurement process of A_t for different values of t, one can define operationally a classical observable $\dot{A}$ with value $\dot{A}(\alpha) = \left(\frac{\partial A_t(\alpha)}{\partial t}\right)_{t=0}$ such that $\hat{\dot{A}} = \dot{\hat{A}}$.

We shall say that $\Sigma = \{S, O, p\}$ is a *classical system* if all its observables are classical. If moreover O and p determine in S a differential structure and a symplectic form η in terms of which the equations of evolution can be expressed in canonical form, a classical system will be called *hamiltonian*.

On the other hand, we shall way that Σ is a *quantum system* if S and O can be put in correspondence, respectively, with the one-dimensional subspaces of a complex Hilbert space H and with the hermitian operators on H, in such a fashion that the function $p(A, \alpha, E)$ can be derived according to the usual rules of quantum mechanics.

Even for quantum systems, and for systems with a more general structure, the derivative $\dot{\hat{A}}$ of the mean value of any observable, if it exists, has a precise operational meaning: its determination can be described as a multiple experiment consisting in the measurement of A_t for different values of the time t, repeated many times for each value of t. However it is not necessarily true that O contains an observable $\dot{A}$ with mean value $\hat{\dot{A}} = \dot{\hat{A}}$.

6. *Classical correspondent and macroscopic analogue of a system.*

On account of the above definitions and remarks, to the most general system $\Sigma = \{S, O, p\}$ one can associate a well-determined classical system $\Sigma_c = \{S_c, O_c, p_c\}$, which will be called the *classical correspondent* of Σ, characterized by the following conditions:

a) every state α of S has a *correspondent* α_c in S_c, and every element of S_c is the correspondent of at least one state of S;

b) every observable A of O has a *correspondent* A_c in O_c, and every element of O_c is the correspondent of at least one observable of O;

c) if α_c and A_c are correspondents of A and α, one has $p_c(A_c, \alpha_c, \hat{A}(\alpha)) = 1$, i.e. the function $p_c(A_c, \beta_c, E)$ has value 1 if $\hat{A}(\alpha) \in E$ and zero otherwise.

Loosely speaking we can therefore say that the system Σ_c is obtained from Σ by replacing each observable A with a classical observable A_c with value equal to the mean value of A, and by performing the identifications which might then be necessary in order to satisfy the conditions I_a) and I_b) of section 4.

On the other hand we shall say that a system $\Sigma_m = \{S_m, O_m, p_m\}$ is a *macroscopic analogue* of the generic system $\Sigma = \{S, O, p\}$ if Σ_m is a classical *hamiltonian* system, and if there exist a map σ of S on S_m and, for a subset O_μ of O, a one-to-one map ω of O_μ onto O_m such that $p_m(\omega(A), \sigma(\alpha), \hat{A}(\alpha)) = 1$. In this case, starting from Σ, one can construct a new system $\Sigma_\mu = \{S_\mu, O_\mu, p\}$, where S_μ is obtained from S by performing the identifications which might become necessary as a consequence of the exclusion of the observables of $O - O_\mu$. Σ_μ will be called the *reduced system* associated with Σ_m, and its classical correspondent will be denoted by $\Sigma_\mu^* = \{S_\mu^*, O_\mu^*, p^*\}$.

Notice that the classical correspondent of a system is always uniquely determined, but its definition does not imply that it necessarily be hamiltonian; while the latter condition is an essential part of the definition of a macroscopic analogue, whenever it exists. The notion of macroscopic analogue will allow us to clarify what is usually meant in Physics by " Correspondence Principle ", a principle which involves on one hand a system (usually " microscopic") described by quantum mechanics, and on the other hand a system (indeed " macroscopic ") described by classical mechanics: any observable [3] of the classical system has a quantum analogue, the converse being not necessarily true, and there might well be no direct operational relation between

(3) Notice that in our definition of a classical hamiltonian system it is not assumed that O_m contain as many observables as are the functions on the phase space S_m, but only sufficiently many in order that the differential and symplectic structures be operationally determined. For example, O_m, might just contain the observables associated with a particular set of canonical coordinates and their time evolutions.

the preparations of states or between the measurements of observables which are " analogous " in the two schemes. Actually the relation pertains to the correspondence (described by the map ω) between the evolution of the mean values of those observables which have a macroscopic analogue on one hand, and the evolution of the values of their macroscopic analogues on the other hand.

7. *The Correspondence Principle.*

Let us draw our attention on a quantum system $\Sigma_q = \{S_q , O_q , p_q\}$, and assume that it possesses a macroscopic analogue $\Sigma_m = \{S_m , O_m , p_m\}$. We shall keep denoting by ω the one-to-one map on O_m of the subset O_μ of O_q constituted by the observables of Σ_q which possess macroscopic analogues.

The commutators of the hermitian representatives of the observables of Σ_q gives rise to a composition law in O_μ, and an analogous law arises in O_m from the Poisson-brachets of the functions representing the observables of Σ_m on the phase-space S_m. The *Correspondence Principle* can be stated as follows:

If a quantum system $\Sigma_q = \{S_q , O_q , p_q\}$ has a macroscopic analogue $\Sigma_m = \{S_m , O_m , p_m\}$, the one-to-one correspondence relating the set O_m and the set O_μ of the observables of Σ_q endowed with a classical analogue is compatible with the composition laws which arise, in these sets, from the classical Poisson-brackets and the quantum commutators, respectively.

We shall now use this precise formulation of the principle to get further insight in its content.

8. *Geometric interpretation.*

We shall say that two states α_1 and α_2 of S_q are " equivalent " if and only if $\hat{A}_\mu (\alpha_1) = \hat{A}_\mu (\alpha_2)$ for any observable A_μ of O_μ. The map σ of S_q on S_m, compatible with this equivalence relation, determines a one-to-one correspondence σ' between the space S^* of the equivalence classes in S_q and the space S_m. Let us denote by A_μ the generic observable of O_μ, by α the generic state of S_q, by α^* the corresponding equivalence class: by means of the maps σ' and ω one can identify Σ_m with the classical correspondent $\Sigma_\mu^* = \{S^*, O_\mu^* , p^*\}$ of Σ_μ, so that $p^* (A_\mu^* , \alpha^*, E) = p_m (\omega (A_\mu) , \sigma' (\alpha^*) , E)$ or, equivalently, $p^* (A_\mu^* , \alpha^* , \hat{A}_\mu (\alpha)) = 1$.

Let us introduce in the classical phase-space $S_m \equiv S^*$ an arbitrary system of local coordinates $x_1 , x_2 , \cdots , x_{2n}$ in some neighbourhood of the point α^*. In S_q the equivalence class α^* is a subset of the projective Hilbert space $\tilde{H}$ associated with the quantum-mechanical representation of Σ_q. If such a subset is a Hilbert submanifold of $\tilde{H}$, we can introduce in it a system of local coordinates $y_1 , y_2 , \cdots$ defined in some neighbourhood of its generic point α. Setting $z_h = x_h , z_{2n+h} = y_h$, in S_q the point α admits a neighbourhood U_α homeomorphic to the topological product of suitable neighbourhoods of α^* in S^* and of α in α^*, in which $z_1 , z_2 , \cdots$ can be regarded as local coordinates.

As shown in part I, the quantum-mechanical structure of Σ_q determines in S_q a skew-symmetric tensor field η with respect to which the Poisson-bracket $\sum_{h,k} \eta^{hk} \frac{\partial \hat{A}}{\partial z^h} \frac{\partial \hat{B}}{\partial z^k}$ of the mean values of any pair of observables A and B of O_q coincides with the mean value of the observable represented by the hermitian operator $-2i(\mathbf{AB}-\mathbf{BA})$, where **A** and **B** denote the hermitian representatives of A and B. If A and B belong to O_μ, their mean values only depend on the x-coordinates, and not on the y's, so that their associated Poisson-bracket is simply $\sum_{h,k}{}_1^{2n} \eta^{hk} \frac{\partial \hat{A}}{\partial x^h} \frac{\partial \hat{B}}{\partial x^k}$ and gives rise, in the set O_μ^* of the classical system Σ^*, to a composition law determined by the skew-field η^* with components η^{hk} $(h, k = , 2, \cdots, 2n)$ in the local coordinates x of S^*. On the other hand S_m, as phase-space of a classical hamiltonian system, possesses itself a skew-symmetric tensor field η_m, and the correspondence principle states that η_m and η^* correspond to each other in the identification of S_m with S^*.

In other words this amounts to realizing that whenever Σ_q admits a macroscopic analogue Σ_m, the classical correspondent Σ_μ^* of the reduced system Σ_μ associated with Σ_m (obtained from Σ_q by eliminating the observables which do not have a macroscopic analogue and by performing, if necessary, the appropriate identification of states) is hamiltonian and structurally identical with Σ_m.

9. *Generalized hamiltonian systems.*

Whether or not the classical correspondent Σ_μ^* of a reduced system $\Sigma_\mu = \{S_\mu, O_\mu, p\}$ of the generic system Σ is hamiltonian can be directly ascertained, operationally, even if it is not assumed that there exists a macroscopic system Σ_m structurally isomorphic with Σ_μ^*. In fact, once the subset O_μ of O which determines Σ_μ has been selected and the space S_μ of equivalence classes in S has been determined accordingly, it is sufficient to adopt, for the preparation of a given state α^* in α_μ^*, the procedure which pertains to any of its representatives α in S, and for the measurement of a given observable A_μ^* (the classical correspondent of $A_\mu \in O_\mu$) the multiple experiment consisting in the repetition of the measurement of A_μ many times, followed by the averaging of the results.

The conclusions of part I imply that the classical correspondent of a quantum system as a whole is hamiltonian (with an infinite-dimensional phase-space, in general). The same is true, of course for any reduced system possessing itself a quantum-mechanical structure.

The hamiltonian character of the classical mechanical systems and of the classical correspondents of the quantum-mechanical ones suggests the characterization of a wider class of physical systems which includes both classical and quantum systems as special cases. Namely, we shall say that Σ is a *generalized hamiltonian system* whenever its classical correspondent is hamiltonian.

It has been shown in [3] and [5] that, for any system Σ whose space of states S can be regarded as a differential manifold, there is defined in S a symmetric tensor-field $\boldsymbol{g}$ of degree 2 associated with a non-negative quadratic form. If moreover Σ is a generalized hamiltonian system, the hamiltonian character of its classical correspondent Σ_c determines a skew-symmetric tensor field η^* of degree 2 in the space δ^* of the classes of elements of δ with identical classical correspondents, via the identification of δ^* with the symplectic space δ_c.

Among the generalized hamiltonian systems, the classical systems are characterized by their identity with their classical correspondents, which implies the degenerate character of the symmetric field $\boldsymbol{g}$. The quantum systems are characterized by the following properties:

a) the quadratic form associated with $\boldsymbol{g}$ is positive-definite;

b) S coincides with S^*, i.e. distinct states always have distinct classical correpondents;

c) the tensor fields $\boldsymbol{g}$ and η constitute a projective Hilbert space structure on S (see Ref. [5]).

Condition *b*) is a consequence of that existence, for any state α of a quantum system, of an observable A_α (represented by the projection operator on the one-dimensional subspace associated with α in H) with mean value equal to 1 on α and less than 1 on every other state.

The content of condition *c*) can presumably be better understood by a further analysis of the relation, pointed out in Ref. [5], between the commutators of the gradients of the mean values of the observables and their Poisson-brackets, all concepts which make sense in the framework of generalized hamiltonian systems independently of the specific postulates of quantum mechanics. We hope to be able to develop this matter in a subsequent paper.

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