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Results and a conjecture on fixed points

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Geometria differenziale. — *Results and a conjecture on fixed points*. Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se S ed T sono applicazioni di uno spazio metrico completo X in sè, con S oppure T continuo, tali che

$$d(Sx, TSy) \leqq c \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), \tfrac{1}{2}[d(x, TSy) + d(Sy, Sx)]\}$$

per tutti gli x, y di X, dove $0 \leqq c < 1$, allora S ed T hanno un unico punto fisso comune. Si ha la congettura che, se

$$d(Sx, TSy) \leqq c \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\},$$

allora S ed T hanno un unico punto fisso comune.

In a recent paper, see [1], the following theorem was proved:

THEOREM 1. *Suppose* S *is a continuous mapping and* T *is a mapping of the complete metric space* X *into itself satisfying the inequality*

$$d(Sx, TSy) \leq ad(x, Sy) + b\{d(x, Sx) + d(Sy, TSy)\} + \\ + c\{d(x, TSy) + d(Sy, Sx)$$

for all x, y *in* X, *where* $a, b, c \geq 0$ *and* $a + 2b + 2c < 1$, *then* S *and* T *have a unique common fixed point.*

We first of all prove the following generalization of Theorem 1:

THEOREM 2. *Suppose* S *and* T *are mappings of the complete metric space* X *into itself, with either* S *or* T *continuous, satisfying the inequality*

$$d(Sx, TSy) \leq c \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), \\ \tfrac{1}{2}[d(x, TSy) + d(Sy, Sx)]\}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* S *and* T *have a unique common fixed point.*

Proof. Let x be an arbitrary point in X. Then

$$d((ST)^n x, T(ST)^n x) \leq c \max\{d(T(ST)^{n-1} x, (ST)^n x), d((ST)^n x, T(ST)^n x), \\ \tfrac{1}{2} d(T(ST)^{n-1} x, T(ST)^n x)\} \leq c \max\{d(T(ST)^{n-1} x, (ST)^n x), \\ d((ST)^n x, T(ST)^n x), \tfrac{1}{2}[d(T(ST)^{n-1} x, (ST)^n x) + \\ + d((ST)^n x, T(ST)^n x)]\} = cd(T(ST)^{n-1} x, (ST)^n x),$$

(*) Nella seduta del 23 giugno 1977.

since $c < 1$. Similarly, we have

$$d(\mathrm{T}(\mathrm{ST})^{n-1} x, (\mathrm{ST})^n x) \leq cd((\mathrm{ST})^{n-1} x, \mathrm{T}(\mathrm{ST})^{n-1} x),$$

and it follows that

$$d((\mathrm{ST})^n x, \mathrm{T}(\mathrm{ST})^n x) \leq c^{2n-1} d(\mathrm{T}x, \mathrm{ST}x)$$

and

$$d(\mathrm{T}(\mathrm{ST})^{n-1} x, (\mathrm{ST})^n x) \leq c^{2n-2} d(\mathrm{T}x, \mathrm{ST}x).$$

Since $c < 1$, it follows that the sequence

$$\{x, \mathrm{T}x, \mathrm{ST}x, \cdots, (\mathrm{ST})^n x, \mathrm{T}(\mathrm{ST})^n x, \cdots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X. Thus

$$\lim_{n\to\infty} (\mathrm{ST})^n x = \lim_{n\to\infty} \mathrm{T}(\mathrm{ST})^n x = z.$$

Now suppose that S is continuous. Then

$$\lim_{n\to\infty} \mathrm{S}[\mathrm{T}(\mathrm{ST})^n x] = \mathrm{S}z = z$$

and so z is a fixed point of S. Further

$$d(z, \mathrm{T}z) = d(\mathrm{S}z, \mathrm{TS}z) \leq c \max\{d(z, \mathrm{S}z), d(\mathrm{S}z, \mathrm{TS}z),$$
$$\tfrac{1}{2} d(z, \mathrm{TS}z)\} = cd(z, \mathrm{T}z)$$

and, since $c < 1$, we see that $\mathrm{T}z = z$. Thus z is a common fixed point of S and T.

Alternatively, let us now suppose that T is continuous. Then

$$\lim_{n\to\infty} \mathrm{T}[(\mathrm{ST})^n x] = \mathrm{T}z = z$$

and so z is a fixed point of T. Further

$$\begin{aligned} d(z, \mathrm{S}z) &\leq d(z, \mathrm{T}(\mathrm{ST})^n x) + d(\mathrm{S}z, \mathrm{T}(\mathrm{ST})^n x) \\ &\leq d(z, \mathrm{T}(\mathrm{ST})^n x) + c \max\{d(z, (\mathrm{ST})^n x), d(z, \mathrm{S}z), \\ &\quad d((\mathrm{ST})^n x, \mathrm{T}(\mathrm{ST})^n x), \tfrac{1}{2}[d(z, \mathrm{T}(\mathrm{ST})^n x) + d((\mathrm{ST})^n x, \mathrm{S}z)]\} \end{aligned}$$

and on letting n tend to infinity we see that

$$d(z, \mathrm{S}z) \leq cd(z, \mathrm{S}z).$$

Since $c < 1$, it now follows that z is again a common fixed point of S and T.

Now suppose that z' is a second common fixed point of S and T. Then

$$d(z, z') = d(Sz, TSz') \leq c \max\{d(z, Sz'), d(z, Sz), d(Sz', TSz'), \tfrac{1}{2}[d(z, TSz') + d(Sz', Sz)]\} = cd(z, z').$$

Since $c < 1$, we see that $z = z'$ and so the common fixed point z is unique. This completes the proof of the theorem.

It was shown in [1] that Theorem 1 would not hold if neither S nor T was continuous. The condition that either S or T be continuous in Theorem 2 is therefore necessary.

On putting $S = T$ in Theorem 2 we have the following

COROLLARY. *Suppose* T *is a continuous mapping of the complete metric space* X *into itself satisfying the inequality*

$$d(Tx, T^2y) \leq c \max\{d(x, Ty), d(x, Tx), d(Ty, T^2y), \tfrac{1}{2}[d(x, T^2y) + d(Ty, Tx)]\}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* T *has a unique fixed point.*

We can however prove the following theorem which is a generalization of this corollary:

THEOREM 3. *Suppose* T *is a continuous mapping of the complete metric space* X *into itself satisfying the inequality*

$$d(Tx, T^2y) \leq c \max\{d(x, Ty), d(x, Tx), d(Ty, T^2y), d(x, T^2y), d(Ty, Tx)\} \tag{1}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* T *has a unique fixed point.*

Proof. Let x be an arbitrary point in X and let us suppose that the sequence $\{T^n x : n = 1, 2, \cdots\}$ is unbounded. Then the sequence $\{d(Tx, T^n x) : n = 1, 2, \cdots\}$ is unbounded and so there must exist an integer $n \geq 2$ such that

$$d(Tx, T^n x) > \frac{c}{1-c} d(Tx, x).$$

We will suppose that this n is the smallest such n so that we will have

$$d(Tx, T^n x) > \frac{c}{1-c} d(Tx, x) \geq \max\{d(Tx, T^r x) : r = 1, 2, \cdots, n-1\}. \tag{2}$$

However, with this n we know that

$$(3) \qquad d(\mathrm{T}x, \mathrm{T}^n x) \leq c \max\{d(x, \mathrm{T}^{n-1} x), d(x, \mathrm{T}x), d(\mathrm{T}^{n-1} x, \mathrm{T}^n x), d(x, \mathrm{T}^n x), d(\mathrm{T}x, \mathrm{T}^{n-1} x)\} \leq c \max\{d(x, \mathrm{T}^{n-1} x), d(\mathrm{T}^{n-1} x, \mathrm{T}^n x), d(x, \mathrm{T}^n x)\}$$

on using inequality (2).

Now

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq cd(x, \mathrm{T}^{n-1} x)$$

implies that

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq c\{d(x, \mathrm{T}x) + d(\mathrm{T}x, \mathrm{T}^{n-1} x)\} < (1-c)\, d(\mathrm{T}x, \mathrm{T}^n x) + cd(\mathrm{T}x, \mathrm{T}^{n-1} x) = d(\mathrm{T}x, \mathrm{T}^n x),$$

using inequality (2), which gives a contradiction.

Next

$$(4) \qquad d(\mathrm{T}x, \mathrm{T}^n x) \leq cd(\mathrm{T}^{n-1} x, \mathrm{T}^n x)$$

implies that on using inequality (1)

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq c^2 \max\{d(\mathrm{T}^{n-2} x, \mathrm{T}^{n-1} x), d(\mathrm{T}^{n-1} x, \mathrm{T}^n x), d(\mathrm{T}^{n-2} x, \mathrm{T}^n x)\} \leq c^3 \max\{d(\mathrm{T}^{n-3} x, \mathrm{T}^{n-2} x), d(\mathrm{T}^{n-2} x, \mathrm{T}^{n-1} x), d(\mathrm{T}^{n-3} x, \mathrm{T}^{n-1} x), d(\mathrm{T}^{n-3} x, \mathrm{T}^n x), d(\mathrm{T}^{n-2} x, \mathrm{T}^n x)\}$$

and so on. Inequality (1) can be used indefinitely, since once we obtain a term of the form $d(\mathrm{T}x, \mathrm{T}^r x)$ with $r = 2, 3, \cdots, n-1$, we can omit it because of inequality (2) and we can omit the term $d(\mathrm{T}x, \mathrm{T}^n x)$, because of the inequality itself. It follows that

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq c^k \max\{d(\mathrm{T}^r x, \mathrm{T}^s x) : r = 1, \cdots, n-1, s = r+1, \cdots, n\}$$

for $k = 1, 2, \cdots$ and so

$$d(\mathrm{T}x, \mathrm{T}^n x) < \frac{c}{1-c}\, d(\mathrm{T}x, x)$$

for large enough k, since $c < 1$ and n is fixed. This contradicts the definition of n and so inequality (4) cannot hold.

Finally

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq cd(x, \mathrm{T}^n x)$$

implies that

$$d(\mathrm{T}x, \mathrm{T}^n x) \leq c\{d(x, \mathrm{T}x) + d(\mathrm{T}x, \mathrm{T}^n x)\}$$

and so

$$d(Tx, T^n x) \leq \frac{c}{1-c} d(x, Tx),$$

which again gives a contradiction.

It follows that inequality (3) cannot hold and this implies that our assumption that the sequence $\{T^n x\}$ is unbounded is false.

Having established that the sequence $\{T^n x\}$ is bounded, let us put

$$M = \sup \{d(T^p x, T^q x) : p, q = 0, 1, 2, \cdots\} < \infty.$$

Then, for arbitrary $\varepsilon > 0$, choose N so that

$$c^N M < \varepsilon.$$

It follows that for $r, n \geq N$

$$d(T^r x, T^n x) \leq c \max \{d(T^{r-1} x, T^{n-1} x), d(T^{r-1} x, T^r x), d(T^{n-1} x, T^n x),$$
$$d(T^{r-1} x, T^n x), d(T^{n-1} x, T^r x)\} \leq c^N M < \varepsilon.$$

Thus $\{T^n x\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Since T is continuous it follows that

$$Tz = z$$

and so z is a fixed point of T. The uniqueness of z follows easily. This completes the proof of the theorem.

The above results suggest that we make the following:

CONJECTURE. *Suppose* S *and* T *are mappings of the complete metric space* X *into itself, with either* S *or* T *continuous, satisfying the inequality*

$$d(Sx, TSy) \leq c \max \{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* S *and* T *have a unique common fixed point.*

We now consider analogous results for compact metric spaces. First of all we have

THEOREM 4. *Suppose* S *and* T *are continuous mappings of the compact metric space* X *into itself satisfying either the inequality*

$$d(Sx, TSy) < \max \{d(x, Sy), d(x, Sx), d(Sy, TSy), \tfrac{1}{2}[d(x, TSy) +$$
$$+ d(Sy, Sx)]\} \quad \textit{if} \quad \max \{d(x, Sy), d(x, Sx), d(Sy, TSy),$$
$$d(x, TSy), d(Sy, Sx)\} \neq 0$$

or the equality

$$d(\mathrm{S}x, \mathrm{TS}y) = 0$$

otherwise, for all x, y *in* X, *then* S *and* T *have a unique common fixed point.*

Proof. First of all suppose there exists $c < 1$ such that

$$d(\mathrm{S}x, \mathrm{TS}y) \leq c \max \{d(x, \mathrm{S}y), d(x, \mathrm{S}x), d(\mathrm{S}y, \mathrm{TS}y), \tfrac{1}{2}[d(x, \mathrm{TS}y) + d(\mathrm{S}y, \mathrm{S}x)]\}$$

for all x, y in X. The result then follows from Theorem 2.

If no such c exists, then if $\{c_n\}$ is a monotonically increasing sequence of real numbers with $\lim_{n\to\infty} c_n = 1$, we can find sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(\mathrm{S}x_n, \mathrm{TS}y_n) > c_n \max \{d(x_n, \mathrm{S}y_n), d(x_n, \mathrm{S}x_n), d(\mathrm{S}y_n, \mathrm{TS}y_n), \tfrac{1}{2}[d(x_n, \mathrm{TS}y_n) + d(\mathrm{S}y_n, \mathrm{S}x_n)]\}$$

for $n = 1, 2, \cdots$. Since X is compact, we can choose convergent subsequences $\{x_{n(r)}\} = \{x'_r\}$ and $\{y_{n(r)}\} = \{y'_r\}$ of $\{x\}$ and $\{y_n\}$ converging to x and y respectively. Then if $\{c_{n(r)}\} = \{c'_r\}$, we have

$$d(\mathrm{S}x'_r, \mathrm{TS}'_r) > c'_r \max \{d(x'_r, \mathrm{S}y'_r), d(x'_r, \mathrm{S}x'_r), d(\mathrm{S}y'_r, \mathrm{TS}y'_r), \tfrac{1}{2}[d(x'_r, \mathrm{TS}y'_r) + d(\mathrm{S}y'_r, \mathrm{S}x'_r)]\}$$

for $r = 1, 2, \cdots$. Letting r tend to infinity we see that

$$d(\mathrm{S}x, \mathrm{TS}y) \geq \max \{d(x, \mathrm{S}y), d(x, \mathrm{S}x), d(\mathrm{S}y, \mathrm{TS}y), \tfrac{1}{2}[d(x, \mathrm{TS}y) + d(\mathrm{S}y, \mathrm{S}x)]\}.$$

This can happen only if

$$x = \mathrm{S}x = \mathrm{S}y = \mathrm{TS}y$$

and this implies that x is a common fixed point of S and T.

Now suppose that x' is a second distinct common fixed point of S and T. Then we have

$$\begin{aligned} d(x, x') &= d(\mathrm{S}x, \mathrm{TS}x') \\ &< \max \{d(x, \mathrm{S}x'), d(x, \mathrm{S}x), d(\mathrm{S}x', \mathrm{TS}x'), \tfrac{1}{2}[d(x, \mathrm{TS}x') + \\ &\quad + d(\mathrm{S}x', \mathrm{S}x)]\} = d(x, x'), \end{aligned}$$

giving a contradiction The common fixed point must therefore be unique. This completes the proof of the theorem.

When $S = T$ we have

THEOREM 5. *Suppose* T *is a continuous mapping of the compact metric space* X *into itself satisfying either the inequality*

$$d(Tx, T^2y) < \max\{d(x, Ty), d(x, Tx), d(Ty, T^2y), d(x, T^2y), d(Ty, Tx)\}$$

if $\max\{d(x, Ty), d(x, Tx), d(Ty, T^2y), d(x, T^2y), d(Ty, Tx)\} \neq 0$

or the equality

$$d(Tx, T^2y) = 0$$

otherwise, for all x, y *in* X, T *then* T *has a unique fixed point.*

The proof of this theorem is omitted, being very similar to that of Theorem 4 apart that Theorem 3 is used in the proof instead of Theorem 2.

We finally have

THEOREM 6. *Suppose* S *and* T *are continuous mappings of the compact metric space* X *into itself satisfying either the inequality*

$$d(Sx, TSy) < \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\}$$

if $\max\{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\} \neq 0$

or the equality

$$d(Sx, TSy) = 0$$

otherwise, for all x, y *in* X. *Suppose further that there exists no* c, *with* $0 \leq c < 1$, *such that*

$$d(Sx, TSy) \leq c \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\}$$

for all x, y *in* X, *then* S *and* T *have a unique common fixed point.*

The proof of this Theorem is omitted. The proof is again very similar to the proof of Theorem 4 but this time the condition of the theorem is used in the proof instead of Theorem 2.

If our conjecture is true, we can of course omit the condition in the Theorem that there exists no $c < 1$ such that

$$d(Sx, TSy) \leq c \max\{d(x, Sy), d(x, Sx), d(Sy, TSy), d(x, TSy), d(Sy, Sx)\}$$

for all x, y in X.

REFERENCE

[1] B. FISHER - *Results on common fixed points*, «Math. Japon.» (to appear).