Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Javed Ahsan

Completely hereditary rings

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **63** (1977), n.3-4, p. 159–163.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_63_3-4_159_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ferie 1977 (Settembre–Ottobre)

(Ogni Nota porta a piè di pagina la data di arrivo o di presentazione)

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Completely hereditary rings.* Nota di JAVED AHSAN, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si fornisce una caratterizzazione di anelli artiniani completamente ereditarii e semi completamente ereditarii.

1. INTRODUCTION

We recall that a ring R is right hereditary in case each of its right ideals is projective. It is well known that R is right hereditary if and only if every submodule of a projective right R-module is projective. Dually, R is right hereditary if and only if each factor module of an injective right R-module is injective. It is, however, not necessary for a hereditary ring to have the property that each homomorphic image of a quasi-injective R-module is quasi-injective. Fuller [6] called a ring R " completely hereditary " if each submodule of a quasi-projective R-module is quasi-projective. Dual to completely hereditary rings are rings for which each homomorphic image of a quasi-injective module is quasi-injective. Though in the most general case it is not known whether the two classes of rings so defined coincide or not, Fuller [6] did prove that the two classes coincide in the Artinian case.

The purpose of this paper is to study some properties of rings for which each homomorphic image of a quasi-injective module is quasi-injective. Following the terminology of Fuller, we shall call such rings " completely here-

(*) Nella seduta del 12 febbraio 1977.

ditary". We shall also briefly study rings for which each homomorphic image of a finitely generated quasi-injective module is again quasi-injective. Such rings will be called "semi-completely hereditary" or, in short, SCH-rings.

2. Preliminaries

Throughout this paper all rings are associative rings with identity and all modules are unitary right modules. An R-module M is quasi-injective if every homomorphism from a submodule of M into M extends to an endomorphism of M. Quasi-projective modules can be defined dually. For an ideal I of R, the quasi-injectivity of M as an R/I-module implies the quasi-injectivity of M as an R-module. Also, if M_R is a quasi-iniective module and I annihilates M, then $M_{R/I}$ is quasi-injective (see Lemma 2 of [1]). If M is quasi-injective, then so is M^n ($=M\oplus\cdots\oplus M$; n times) (Proposition 2.4, Harada [7]). Every faithful quasi-injective module over an Artinian ring is injective (Theorem 1.2, Fuller [5]). A ring R is called a V-ring if each simple R-module is injective.

3. Main Results

An interesting characterization of hereditary rings due to Matlis [9] states that a ring R is right hereditary if and only if the sum of every pair of injective submodules in a right R-module is injective. One may wonder whether a corresponding characterization of completely hereditary rings can be found in the quasi-injective setting. We do not know the answer in general but we shall prove that such a characterization does exist in the Artianian case. First we add a remark which we borrow from Faith [2, p. 63].

Let R be a ring and E an R-module. Suppose N is a submodule of E and $Q = E\oplus E$ be the direct product (or direct sum) of two copies of E. Let $K = \{(x , x) \in Q \mid x \in N\}$ and $\mathbf{Q} = Q/K$.

Define $M_1 = \{y + K \in \mathbf{Q} \mid y \in (E , 0)\}$ and $M_2 = \{y + K \in \mathbf{Q} \mid y \in (0 , E)\}$. Then $\mathbf{Q} = M_1 + M_2$; $M_i \simeq E$ $(i = 1 , 2)$ and $M_1 \cap M_2 \simeq N$.

We now prove the following lemma.

Lemma 1. *Let* R *be an Artinian ring. If each sum* $M_1 + M_2$ *of quasi-injective submodules of an* R*-module is quasi-injective, then* R *is completely hereditary.*

Proof. Let E_R be a quasi-injective module and N_R be a submodule of E_R. In order to prove the lemma, we show that $(E/N)_R$ is (R–) quasi-injective. We do this as follows:

Let $I = \text{ann}_R(E)$ and write $\mathbf{R} = R/I$.

The $E_{\mathbf{R}}$ is a faithful quasi-injective module. Since $\mathbf{R}$ is an Artinian ring, $E_{\mathbf{R}}$ is injective.

Let us now consider E_R and N_R a submodule of E_R. Write $Q_R = E_R \oplus E_R$ to be the direct product or direct sum of two copies of E_R and let $K_R = \{(x, x) \in Q_R \mid x \in N_R\}$ and $\mathbf{Q}_R = Q/K$.

Define $M^1_R = \{y + K \in \mathbf{Q}_R \mid y \in (E, o)\}$ and $M^2_R = \{y + K \in \mathbf{Q}_R \mid y \in (o, E)\}$. Then, in view of the remark above,

$$\mathbf{Q}_R = M^1_R + M^2_R,$$
$$M^i_R \cong E_R \qquad (i = 1, 2), \quad \text{and}$$
$$M^1_R \cap M^2_R \cong N_R.$$

Since E_R is an injective module, M^i_R $(i = 1, 2)$ are injective. Also, since M^1_R is injective, it follows that M^1_R is quasi-injective. Similarly, M^2_R is quasi-injective. Therefore, by the assumption, $(M_1 + M_2)_R$ is **R**-quasi-injective. Then $(M_1 + M_2)_R$ is (**R**–) quasi-injective. Therefore, $\mathbf{Q}_R = M^1_R + M^2_R$ is (**R**–) quasi-injective. But M^1_R is injective, hence there exists a submodule G_R of $\mathbf{Q}_R$ such that $\mathbf{Q}_R = M^1_R \oplus G_R$. Therefore, G_R is quasi-injective. Now $G_R \cong (M_1 + M_2)/M_1 \cong (M_2/M_1 \cap M_2)_R$. Since $M^2_R \cong E_R$ and $(M_1 \cap M_2) \cong N_R$, it follows that $(E/N)_R \cong G_R$. Hnce, $(E/N)_R$ is quasi-injective. This implies that $(E/N)_R$ is R–quasi-injective. This proves the lemma.

THEOREM 2. *Let* R *be an Artinian ring. Then the following statements are equivalent*:

(1) *The sum of every pair of isomorphic quasi-injective submodules in any right* R-*module is quasi-injective.*

(2) R *is completely hereditary.*

Proof. (1) ⇒ (2)

This can be proved by repeating the arguments of the above Lemma.

(2) ⇒ (1).

Let M_1 and M_2 be any two isomorphic quasi-injective submodules of an R-module. Then $M_1 + M_2$ is a homomorphic image of $M_1 \oplus M_2$. Since $M_1 \oplus M_2$ is quasi-injective (Proposition 2.4, Harada [7]) and R is completely hereditary, $M_1 + M_2$ is quasi-injective.

If we assume that the sum of any two quasi-injectives is quasi-injective, then we obtain a result which may be of independent interest. First, we obtain the following lemma.

LEMMA 3. *Let* R *be any ring. If a sum* $M_1 + M_2$ *of any two quasi-injectives is quasi-injective, then* R *is a right Noetherian right hereditary* V-*ring and every quasi-injective is injective.*

Proof. We first prove that every quasi-injective is injective. Let M be any quasi-injective module and write $A = E(R) \oplus M$; where $E(R)$ is the injective envelope of R_R. Then, by our assumption, A is quasi-injective. Since R ⊆ A and A is quasi-injective, any map $f : I \to A$; I a right ideal of R;

extends to a map $f' : R \to A$. Hence, by Bear's Criterion, A is injective and so M is injective. Since every semi-simple Artinian module is quasi-injective (Fait [2], Cor. 9 p. 55), every such module is injective and, hence, R is right Noetherian (Kurshan [8], Theorem 2.4). Also, since every simple module is quasi-injective, every simple module is injective. Hence, R is a V-ring. If M_1 and M_2 are injective submodules of an R-module, then $M_1 + M_2$ is quasi-injective by the assumption, and so $M_1 + M_2$ is injective. Therefore, R is a hereditary ring.

We now prove the following theorem.

THEOREM 4. *Let* R *be a commutative ring. Then the following statements are equivalent*:

(1) *Each ordinary sum of quasi-injective modules is quasi-injective.*

(2) R *is semi-simple Artinian.*

Proof. 1. Suppose each sum of quasi-injectives is quasi-injective. Then R is a Noetherian V-ring by the above lemma. Therefore, R is a direct product of simple V-rings (Faith [3]). Since R is a commutative ring, each simple ring is a field. Therefore, R is semi-simple Artinian.

2. The converse is immediate.

We shall call a ring R "semi-completely hereditary" or, in short, an "SCH-ring" in case each homomorphic image of a finitely generated quasi-injective R-module is quasi-injective. We hall obtain characterization of SCH-rings in the commutative case. First we prove a lemma.

LEMMA 5. *Let* R *be a commutative ring. Then every faithfull finitely generated quasi-injective module is injective.*

Proof. Let M be a faithful finitely generated quasi-injective module. Then, by Proposition 2.28 on page 146 of Faith [4], M is compact faithful in the sense that $R \subseteq M^n$; for a finite integer $n > 0$. Since M is quasi-injective, so is M^n. Hence, any map $f : I \to M^n$ (I an ideal of R) extends to a map $f' : R \to M^n$. This implies, by Baer's Criterion, that M^n is injective, so M is injective.

THEOREM 6. *Let* R *be a commutative ring. Then the following statements are equivalent*:

(1) *Each sum of a pair of finitely generated isomorphic quasi-injective submodules of an* R*-module is quasi-injective.*

(2) R *is an* SCH*-ring.*

Proof. 1. (1) $\Rightarrow$ (2).

Let E_R be a finitely generated quasi-injective module and N_R be a submodule of E_R. In order to prove that R is an SCH-ring, we must show that $(E/N)_R$ is (R–) quasi-injective.

Let $I = \text{ann}(E)$ and write $\mathbf{R} = R/I$. Then $E_{\mathbf{R}}$ is a faithful finitely generated quasi-injective **R**–module. Hence, by the above lemma, $E_{\mathbf{R}}$ is (**R**–) injective. Now, by using the arguments employed in the proof of Lemma 1, we can show that $(E/N)_R$ is (R–) quasi-injective.

2. $(2) \Rightarrow (1)$.

Let M_1 and M_2 be any two finitely generated isomorphic quasi-injective submodules of an R-module, then $M_1 \oplus M_2$ is finitely generated and quasi-injective. Since $M_1 + M_2$ is a homomorphic image of $M_1 \oplus M_2$, and R is an SCH-ring, it follows that $M_1 + M_2$ is quasi-injective. This proves the theorem.

REFERENCES

[1] J. AHSAN (1973) – *Rings all of whose cyclic modules are quasi-injective*, « Proc. London Math. Soc. », *27* (3), 425–439.

[2] C. FAITH (1967) – *Lectures on Injective Modules and Quotient Rings*, Springer–Verlag Lecture Notes, New York.

[3] C. FAITH (1972) – *Moduels Finite Over Endomorphism Rings, Lectures on Rings and Modules* (Tulane Univ. Ring Theory Conference Proceedings) « Pringer–Verlag Lecture Notes », *246*, 145–189.

[4] C. FAITH (1973) – *Algebra – Rings, Modules and Categories*, « I, Springer–Verlag Lecture Notes », New York.

[5] K. R. FULLER (1969) – *On direct representations of quasi-injectives and quasi-projectives*, « Arch. Math. », *20*, 495–502.

[6] K. R. FULLER (1972) – *Relative projectivity and injectivity classes determined by simple modules*, « J. London Math. Soc. », 5.

[7] M. HARADA (1965) – *Note on quasi-injective modules*, « Osaka J. Math. », *2*, 351–356.

[8] R. P. KURSHAN (1970) – *Rings whose cyclic modules have finitely generated socle*, « J. Algebra », *15*, 376–386.

[9] E. MATLIS (1958) – *Injective Modules Over Noetherian Rings*, « Pac. J. Math. », *8*, 511–528.