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Age-dependent population dynamics

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Analisi matematica. — *Age-dependent population dynamics.* Nota [(*)] di GABRIELLA DI BLASIO [(**)] e LAMBERTO LAMBERTI [(**)], presentata dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si studia il problema di Cauchy per una equazione differenziale derivante dallo studio della diffusione di una singola specie biologica. Si dimostra l'esistenza e l'unicità della soluzione di tale problema e la dipendenza continua dai dati.

1. INTRODUCTION

This paper is concerned with the study of the following partial differential equation

$$(1)\qquad \frac{\partial}{\partial t} u(t, a, x) + \frac{\partial}{\partial a} u(t, a, x) = -\mu(a) u + \int_0^{+\infty} k(a, a') \Delta u \, da'$$

$$t, a \geq 0 \quad , \quad x \in \Omega$$

which has been proposed by Gurtin as a model for diffusion of a single species population [4].

Here $u(t, a, x)$ represents the density per unit volume and age of some biological population at time t at the location x in Ω and μ is the age-dependent mortality rate so that $-\mu u$ represents the death process.

We shall study equation (1) together with the following additional conditions

(i) an initial space and age distribution $u(0, a, x) = u_0(a, x)$;

(ii) an age boundary condition representing the birth process $u(t, 0, x) = b(t, x)$;

(iii) a spatial boundary condition $\frac{\partial}{\partial \nu} u = 0$ where $\frac{\partial}{\partial \nu}$ is the exterior normal derivative at the boundary $\partial\Omega$.

2. PRELIMINARIES

In this section we collect some known results concerning dissipative functions (see [2] and [3]).

Let H be a real Hilbert space; a function $A : D(A) \subseteq H \to H$ is said to be *dissipative* if for each $u, v \in D(A)$ we have $(Au - Av, u - v) \leq 0$. A dissipative A is said to be *hyper-dissipative* if $(\lambda I - A)(D(A)) = H$ for each $\lambda > 0$.

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If A is hyper-dissipative we can consider the Yosida approximating functions $A_n : H \to H$, $(n \in N)$-defined by

$$A_n = n\,(nI - A)^{-1}\,n - nI = A\,(nI - A)^{-1}\,n\,.$$

The following Lemma collects some known properties of the functions A_n.

LEMMA 1. *If* A *is hyper-dissipative then*:

(i) A_n *is a Lipschitz continuous function*;

(ii) A_n *is hyper-dissipative*;

(iii) *if* u_n *is such that* $u_n \to u$ *and* $A_n u_n \rightharpoonup v$ *(weak convergence) then* $u \in D(A)$ *and* $Au = v$.

3. PROPERTIES OF THE FUNCTIONS $\frac{\partial}{\partial a} + \mu I$ AND $\int k\Delta$

Let $\mu : [0, +\infty[\to R$, $a \to \mu(a) \geq 0$ and $K : [0, +\infty[\times[0, +\infty[\to R$, $(a, a') \to K(a, a')$ be measurable functions; we shall study equation (1) under the following hypotheses

(m_1) $\mu \in L^1_{loc}([0, +\infty[)$

(k_1) there exists a constant c_1 such that for each $u \in L^2(]0, +\infty[)$ we have

$$\left(\int_0^{+\infty}\left(\int_0^{+\infty} K(a, a')\,u(a')\,da'\right)^2 da\right)^{\frac12} \leq c_1\left(\int_0^{+\infty} u^2\,da\right)^{\frac12}$$

(k_2) for each $u \in L^2([0, +\infty[)$ we have

$$\int_0^{+\infty}\int_0^{+\infty} K(a, a')\,u(a)\,u(a')\,da'\,da \geq 0\,.$$

Now let $\Omega \subseteq R^m$ be an open bounded set with smooth boundary $\partial\Omega$ and set $H = L^2(]0, +\infty[\times\Omega)$; we denote by A_β, T and B the operators defined by

$$\begin{cases} D(A_\beta) = \left\{ \begin{array}{l} u \in H,\ a \to u(a, x) \in W^{1,2}([0, +\infty[) \quad \text{and} \quad u(0, x) = \beta \\ \text{for a.e. } x \in \Omega;\ (a, x) \to \dfrac{\partial}{\partial a} u(a, x) \in H \end{array} \right\} \\ A_\beta u = -\dfrac{\partial}{\partial a} u \end{cases}$$

where $\beta \in L^2(\Omega)$ is a given function and

$$\begin{cases} D(T) = H \\ Tu = \displaystyle\int_0^{+\infty} K(a, a')\,u(a', x)\,da' \end{cases}$$

$$\begin{cases} D(B) = \left\{ \begin{array}{l} u \in H\,;\quad \text{for a.e. } a \in [0\,, +\infty[\, x \to u(a\,,x) \in W^{2,2}(\Omega) \\ \text{and } \dfrac{\partial}{\partial\nu} u = 0 \quad \text{for a.e. } x \in \partial\Omega\ ;\ (a\,,x) \to \Delta u \in H \end{array} \right\} \\ Bu \quad = \Delta u. \end{cases}$$

The following theorems are well known

THEOREM 2. $A_\beta - \mu I$ *is hyper-dissipative and we have* $(A_\beta u - \mu u\,, u) \leq \leq \frac{1}{2} \int \beta^2 \, dx$.

THEOREM 3. B *is hyper-dissipative.*

The following Lemmas collect some further properties of A_β, T and B.

LEMMA 2. *The operator* TB *is dissipative.*

Proof. Let $u \in D(B)$ from (k_2) we have

$$(TBu\,, u) = -\int_\Omega \int_0^{+\infty} \int_0^{+\infty} K(a\,, a') \sum_{i=1}^m u_{x_i}(a', x)\, u_{x_i}(a\,, x)\, da'\, da\, dx \leq 0\,.$$

LEMMA 3. *Let* $\beta \in D(B)$ *then for each* $u \in D(A_\beta - \mu I) \cap D(B)$ *we have*

(i) $$-(A_\beta u - \mu u\,, Bu) \leq \frac{1}{2} \int_\Omega |\nabla\beta|^2 \, dx$$

where $|\nabla\beta| = \left(\sum_{i=1}^m \beta_{x_i}^2\right)^{\frac{1}{2}}$.

Proof. It suffices to prove (i) with $u \in C_0^\infty(\Omega \times]0\,, +\infty[)$. We have

$$-(A_\beta u - \mu u\,, Bu) = \int_0^{+\infty} \int_\Omega \left(\frac{\partial u}{\partial a} + \mu u\right) \Delta u \, dx\, da \leq$$

$$-\int_0^{+\infty} \int_\Omega \sum_{i=1}^m \left(\frac{\partial}{\partial a} u_{x_i}\right) u_{x_i} \, dx\, da \leq \frac{1}{2} \int_\Omega |\nabla_\beta|^2 \, dx\,.$$

LEMMA 4. *The operator* TB_n *is continuous and dissipative.*

Proof. The first assertion follows from Lemma 1 (i) and condition (k_1). The second assertion follows from Lemma 2, condition (k_2) and the identity

$$(TB_n u\,, u) = (TB(nI - B)^{-1} nu\,, (nI - B)^{-1} nu) - 1/n\,(TB_n u\,, B_n u)\,.$$

Finally the following Lemma is a consequence of Lemma 3 and Theorem 2.

LEMMA 5. *For each* $u \in D(A_\beta - \mu I)$ *we have*

$$-(A_\beta u - \mu u\,, B_n u) \leq \frac{1}{2} \int_\Omega |\nabla(nI - B)^{-1} n\beta|^2 \, dx + 1/2\,n \int_\Omega |B_n \beta|^2 \, dx\,.$$

4. Existence and uniqueness results

For each $\varepsilon \geq 0$ and $n \in \mathrm{N}$ we shall consider the following regularized problem

$$(2)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = -\mu u + \displaystyle\int_0^{+\infty} \mathrm{K}(a, a')\, \Delta_n u \, \mathrm{d}a' + \varepsilon \Delta_n u + w \\ u(t, 0, x) = b(t, x) \\ u(0, a, x) = u_0(a, x) \end{cases}$$

$$w \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{H}) \quad , \quad b \in \mathrm{C}(0, \mathrm{T}; \mathrm{L}^2(\Omega)) \quad , \quad u_0 \in \mathrm{H} \text{ [1]}$$

where $\Delta_n u = \mathrm{B}_n u(t)$ with B_n defined as in section 3.

To study problem (2) it is convenient to introduce the following definition.

We say that a function $u \in \mathrm{C}(0, \mathrm{T}; \mathrm{H})$ is a strong solution of (2) if there exists $\{u_k\}$ such that

(s_1) $u_k \in \mathrm{W}^{1,2}(0, \mathrm{T}; \mathrm{H})$; $\mu u_k \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{H})$; $a \to u_k(t, a, x) \in$ $\in \mathrm{W}^{1,2}(]0, +\infty[)$ for a.e. $(t, x) \in [0, \mathrm{T}] \times \Omega$ and $t \to \dfrac{\partial}{\partial a} u \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{H})$

(s_2) we have $u_k \to u$ in $\mathrm{C}(0, \mathrm{T}; \mathrm{H})$

$$\frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial a} + \mu u_k \to \int_0^{+\infty} \mathrm{K} \Delta_n u \mathrm{d}a' + \varepsilon \Delta_n u + w \quad \text{in} \quad \mathrm{L}^2(0, \mathrm{T}; \mathrm{H})$$

$$u_k(t, 0, x) \to b(t, x) \quad \text{in} \quad \mathrm{C}(0, \mathrm{T}; \mathrm{L}^2(\Omega))$$

$$u_k(0, a, x) \to u_0(a, x) \quad \text{in} \quad \mathrm{H}.$$

It is not difficult to prove the following result

LEMMA 6. *Let* $\mathrm{K} = 0$, $\varepsilon = 0$, $w = 0$ *and* $u_0 = 0$ *then there exists a unique strong solution* u_1 *of* (2) *and we have*

$$u_1(t, a, x) = \begin{cases} b(t - a, x) \exp\left(\displaystyle\int_0^a \mu(\sigma)\, \mathrm{d}\sigma\right) & t > a \\ 0 & t \leq a. \end{cases}$$

(1) If E is a Hilbert space we denote by $\mathrm{C}(0, \mathrm{T}; \mathrm{E})$ the Banach space of all continuous functions $u : [0, \mathrm{T}] \to \mathrm{E}$; by $\mathrm{L}^2(0, \mathrm{T}; \mathrm{E})$ the Hilbert space of square integrable functions $u : [0, \mathrm{T}] \to \mathrm{E}$ and by $\mathrm{W}^{1,2}(0, \mathrm{T}; \mathrm{E})$ the space of all absolutely continuous functions $u : [0, \mathrm{T}] \to \mathrm{E}$ such that $(\mathrm{d}/\mathrm{d}t)\, u \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{E})$.

LEMMA 7. *Let* $b = 0$ *and* $w = \int_0^{+\infty} K(a, a') \Delta_n u_1 \, da' + \varepsilon \Delta_n u_1$ *then there exists a unique* $\tilde{u}_{\varepsilon,n}$ *strong solution of* (2).

By Theorem 2 and Lemma 4 we have that $A_0 - \mu I$ is hyper-dissipative and that $TB_n + \varepsilon B_n$ is continuous and dissipative so that (see [1, Theorem 1]) $A_0 - \mu I + TB_n + \varepsilon B_n$ is hyper-dissipative and the result follows (see [2] and [3]).

Summarizing we have

THEOREM 4. *Let* u_1 *and* $\tilde{u}_{\varepsilon,n}$ *be the functions defined as in Lemmas* 6, 7; *then the function* $u_{\varepsilon,n} = u_1 + \tilde{u}_{\varepsilon,n}$ *is a strong solution of* (2) *with* $w = 0$.

The following Lemma collects some a-priori estimates for the solutions of (2).

LEMMA 8. *Let* $u_0 \in D(B)$ *and let* $u_{\varepsilon,n}$ *be the strong solution of* (2) *given by Theorem 4. Then for each* $t \in [0, T]$ *we have*:

(i) $$\| u_{\varepsilon,n}(t) \|_H \le \| u_0 \|_H + \| b \|_{C(0,T;L^2(\Omega))}$$

(ii) $$\varepsilon \int_0^t \| \Delta_n u_{\varepsilon,n} \|^2 \, ds \le \tilde{C}(T, u_0, b)$$

where $\tilde{C}(T, u_0, b)$ *is a constant depending on* T, u_0 *and* b.

Moreover if $u_0, \tilde{u}_0 \in D(B)$, $b, \bar{b} \in C(0, T; L^2(\Omega))$ *and if* $u, \tilde{u}_{\varepsilon,n}$ *are the corresponding strong solutions then*

(iii) $$\| u_{\varepsilon,n}(t) - \tilde{u}_{\varepsilon,n}(t) \|_H \le \| u_0 - \tilde{u}_0 \|_H + \| b - \bar{b} \|_{C(0,T;L^2(\Omega))} .$$

Proof. To prove (i) it suffices to take the scalar product of (s_2) with u_k, integrate over $[0, t]$ and use Theorem 2. Assertion (ii) follows from Lemma 5 by taking the scalar product of (s_2) with $\Delta_n u_k$. Finally the proof of (iii) is similar to that of (i).

Finally using Lemma 8 (ii) and Lemma 1 (iii) we get the following existence result for the solutions of (2) in the generalized sense specified below

THEOREM 5. *Let* $u_0 \in D(B)$ *and* $b \in C(0, T; L^2(\Omega))$ *then there exists* u *and* $\{u_\varepsilon\}$ *verifying the following properties*

(g_1) $x \to u_\varepsilon(t, a, x) \in W^{2,2}(\Omega)$ *for a.e.* $(t, a) \in [0, T] \times [0, +\infty[$ *and* $\frac{\partial}{\partial \nu} u = 0$ *a.e.* $x \in \partial\Omega$

(g_2) u_ε *is the strong solution of the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + \mu u = \displaystyle\int_0^{+\infty} \mathrm{K}(a, a')\, \Delta u_\varepsilon \, \mathrm{d}a' + \varphi_\varepsilon \\ u(t, 0, x) = b(t, x) \\ u(0, a, x) = u_0(a, x) \end{cases}$$

(g_3) *we have* $u_\varepsilon \to u$ *in* $\mathrm{C}(0, \mathrm{T}; \mathrm{H})$ *and* $\varphi_\varepsilon \to 0$ *in* $\mathrm{L}^2(0, \mathrm{T}; \mathrm{H})$.

Moreover if $u_0, \bar{u}_0 \in \mathrm{D}(\mathrm{B})$, $b, \bar{b} \in \mathrm{C}(0, \mathrm{T}; \mathrm{L}^2(\Omega))$ and $u, \bar{u}$ are the corresponding generalized solutions then

$$\| u(t) - \bar{u}(t) \|_{\mathrm{H}} \leq \| u_0 - \bar{u}_0 \|_{\mathrm{H}} + \| b - \bar{b} \|_{\mathrm{C}(0, \mathrm{T}; \mathrm{L}^2(\Omega))} .$$

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