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## Further results on the existence of periodic solutions of a certain third order differential equation

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**Equazioni differenziali ordinarie.** — *Further results on the existence of periodic solutions of a certain third order differential equation.* Nota di JAMES O. C. EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano due teoremi di esistenza di soluzioni periodiche dell'equazione differenziale ordinaria del terzo ordine

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + f(x) = p(t)$$

con $p(t)$ funzione periodica.

1. Consider the third order differential equation

$$\dddot{x} + a\ddot{x} + \phi(x)\dot{x} + f(x) = p(t) \tag{1.1}$$

in which a is a constant and $\phi, f, p$ are continuous functions depending only on the arguments shown and $p$ is $\omega$-periodic in $t$, that is $p(t+\omega) = p(t)$ for some $\omega > 0$. Let $\Phi \equiv \int_0^x \phi(\xi)\,d\xi$. There is a result in [1] by Reissig which shows that if the following conditions hold:

(i) $a \neq 0$, (ii) $|x|^{-1}|f(x)| \to 0$ as $|x| \to \infty$, (iii) $f(x)\operatorname{sgn} x \geq 0$ $(|x| \geq 1)$, (iv) $|x|^{-1}|\Phi(x)| \to 0$ as $|x| \to \infty$ and (v) $\int_0^\omega p(t)\,dt = 0$,

then (1.1) has at least one $\omega$-periodic solution. The restrictions (i) and (iv) here were removed in a subsequent paper [2].

We propose, in the present paper, to examine the above result with the following weaker conditions of $f, \phi$ in place of Reissig's (ii) and (iv) respectively:

$$|f(x)| \leq A_1|x| + A_2, \tag{1.2}$$

$$|\Phi(x)| \leq B_1|x| + B_2, \tag{1.3}$$

for all $x$, where $A_i \geq 0$, $B_i \geq 0$ $(i = 1, 2)$ are constants with $A_1$, $B_1$ sufficiently small. The investigation will, furthermore, be concerned with the more general equation

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + f(x) = p(t) \tag{1.4}$$

(*) Nella seduta del 14 gennaio 1978.

in which the coefficient $\psi$, not necessarily constant, is a continuous function depending only on $\dot{x}$: but our other main objective is to identify certain equations (1.4) for which, subject to the conditions ((iii) and (v) above):

$$f(x)\operatorname{sgn} x \geq 0 \qquad (|x| \geq 1) \tag{1.5}$$

$$\int_0^\omega p(t)\,\mathrm{d}t = 0 \tag{1.6}$$

the use of *just one* (*only*) of (1.2) or (1.3) would suffice for the existence of an $\omega$-periodic solution. The position is summed up more clearly in the following two theorems for (1.4) which will be proved shortly:

THEOREM 1. *Given the equation* (1.4) *suppose that* $\phi$, $f$ *and* $p$ *are subject to* (1.3), (1.5) *and* (1.6) *respectively. Then there exists a constant* $\varepsilon_0 > 0$ *such that if* $B_1 \leq \varepsilon_0$, *then* (1.4) *admits of at least one* $\omega$*-periodic solution for all arbitrary* $\psi(\dot{x})$.

Note here the absence of a restriction on $\psi$.

The next theorem covers the special case corresponding to $a \neq 0$ when results are specialized to (1.1).

THEOREM 2. *Given the equation* (1.4) *in which* $p$ *is subject, as before, to* (1.6), *suppose that* $f$ *is subject to* (1.2) *and* (1.5) *and that*

$$\psi(y) \geq \alpha > 0 \qquad \text{for all } y \tag{1.7}$$

*or, otherwise, that*

$$\psi(y) \leq \beta < 0 \qquad \text{for all } y, \tag{1.8}$$

*for some constants* $\alpha$, $\beta$. *Then there exists a constant* $\varepsilon_1 > 0$ *such that if* $A_1 \leq \varepsilon_1$ *then* (1.4) *admits of an* $\omega$*-periodic solution for all arbitrary* $\phi(x)$.

Observe that, when specialized to the case $\psi \equiv$ constant with $f$ bounded Theorem 2 here gives a significant improvement on the results in [2], [3] and [4] for the same equation.

2. The method of proof of either theorem will be by the Leray-Schauder technique, just as in [1] except that for our purpose it will be convenient here to consider the parameter-dependent equation in the form:

$$\dddot{x} + \mu\psi(\dot{x})\,\ddot{x} + \mu\phi(x)\,\dot{x} + (1-\mu)\,c_1 x + \mu f(x) = \mu p(t) \tag{2.1}$$

for dealing with Theorem 1, and in the form:

$$\dddot{x} + \{(1-\alpha)\,\mu + \mu\psi(\dot{x})\}\,\ddot{x} + \mu\phi(x)\,\dot{x} + (1-\mu)\,c_2 x + \mu f(x) = \mu p(t) \tag{2.2}$$

for dealing with Theorem 2 when $\psi$ is subject to (1.7). The case when $\psi$ is subject to (1.8) can also be handled with the same (2.2) but with $\alpha$ replaced by $(-\beta)$ as will be explained in § 6. Here in (2.1) $c_1$ is an arbitrarily chosen, but fixed positive constant. The constant $c_2$ in (2.2) is also positive, but its value is to be fixed (sufficiently small) to advantage later (see (6.4)).

The equations (2.1) and (2.2) reduce to the same (1.4) when $\mu = 1$ and to the constant-coefficient equations:

$$\ddot{x} + c_1 x = 0 \tag{2.3}$$

$$\dddot{x} + \alpha\ddot{x} + c_2 x = 0 \tag{2.4}$$

when $\mu = 0$. It is easily verified that neither of the auxiliary equations corresponding to (2.3) or (2.4) has a purely imaginary root. Thus it will now be sufficient, as in [1], for our proof of Theorem 1 or Theorem 2 with $\psi$ subject to (1.7) to establish merely that there is fixed constant $D > 0$, whose magnitude is *independent of* $\mu$, such that any $\omega$-periodic solution $x(t)$ of (2.1) or (2.2), corresponding to $0 < \mu < 1$ satisfies:

$$|x(t)| \leq D, \ |\dot{x}(t)| \leq D \quad \text{and} \quad |\ddot{x}(t)| \leq D \qquad (\tau \leq t \leq \tau + \omega) \tag{2.5}$$

for some $\tau$.

3. NOTATION. Let $A_3 \equiv \max\limits_{0 \leq t \leq \omega} |p(t)|$. In what follows here the capitals $D, D_0, D_1 \cdots$ are finite positive constants whose magnitudes are independent of the parameter $\mu$ and, indeed, in the context of Theorem 1 depend only on $c_1, A_3, B_2, \phi, \psi$ and $f$, and, in the context of Theorem 2, on $c_2, A_3, A_2, \phi, \psi$ and $f$. The D's without suffixes attached are not necessarily the same in each place of occurrence but the numbered D's: $D_0, D_1, \cdots$ retain a fixed identity throughout.

4. SOME PRELIMINARY RESULTS. As we shall be dealing extensively here with integrals such as $\int x^2 \, dt, \int \dot{x}^2 \, dt, \int \ddot{x}^2 \, dt$ taken over time intervals of length $\omega$, we might as well note that if $x$ is $\omega$-periodic the $\int_\tau^{\tau+\omega} x^2 \, dt = \int_{\tau_0}^{\tau_0+\omega} x^2 \, dt$ for arbitrary $\tau$ and $\tau_0$, since either integral equals $\int_0^\omega x^2 \, dt$ if $x$ is $\omega$-periodic. The same is true of $\int_\tau^{\tau+\omega} \dot{x}^2 \, dt$ and $\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt$.

We shall require specially the use of the following two subsidiary results:

LEMMA 1. *If $x = x(t)$ is continuous and $\omega$-periodic in $t$ then*

$$\int_\tau^{\tau+\omega} \dot{x}^2 \, dt \leq \frac{1}{4} \omega^2 \pi^{-2} \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt . \tag{4.1}$$

*Proof of Lemma.* Let $x$ have the Fourier expansion:

$$x \sim \sum_{r=0}^{\infty} (a_r \cos 2\pi\omega^{-1} rt + b_r \sin 2\pi\omega^{-1} rt), \tag{4.2}$$

so that $\dot{x}$ and $\ddot{x}$ in turn have the corresponding expansions:

$$\dot{x} \sim 2\pi\omega^{-1} \sum_{r=1}^{\infty} -r\{a_r \sin(2\pi\omega^{-1} rt) - b_r \cos(2\pi\omega^{-1} rt)\}$$

$$\ddot{x} \sim 4\pi^2\omega^{-2} \sum_{r=1}^{\infty} r^2\{a_r \cos(2\pi\omega^{-1} rt) + b_r \sin(2\pi\omega^{-1} rt)\}.$$

We have, in the usual manner, from the expansion for $\dot{x}$ that

$$\int_{\tau}^{\tau+\omega} \dot{x}^2\, dt = 2\pi^2\omega^{-1} \sum_{r=1}^{\infty} r^2 (a_r^2 + b_r^2) \tag{4.3}$$

and from the expansion for $\ddot{x}$ that

$$\begin{aligned} \int_{\tau}^{\tau+\omega} \ddot{x}^2\, dt &= 8\pi^4\omega^{-3} \sum_{r=1}^{\infty} r^4 (a_r^2 + b_r^2) \\ &\geq 8\pi^4\omega^{-3} \sum_{r=1}^{\infty} r^2 (a_r^2 + b_r^2) \\ &\geq 4\pi^2\omega^{-2} \int_{\tau}^{\tau+\omega} \dot{x}^2\, dt \end{aligned}$$

by (4.3), which proves (4.1).

LEMMA 2. *Let $x = x(t)$ be an $\omega$-periodic solution of* (2.1) *or of* (2.2) *corresponding to* $0 < \mu < 1$. *Then*

$$\int_{\tau}^{\tau+\omega} x^2\, dt \leq D_0^2 + D_1^2 \int_{\tau}^{\tau+\omega} \dot{x}^2\, dt \tag{4.4}$$

*for some* $D_0, D_1$.

*Proof of Lemma.* Let $x$ have the Fourier expansion (4.2) so that then

$$\int_{\tau}^{\tau+\omega} x^2\, dt = a_0^2 + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) \tag{4.5}$$

whether or not $x$ is a solution of (2.1) or of (2.2).

If in particular $x(t)$ is a solution of (2.1) or of (2.2) then we have, in view of (1.6) on integrating (2.1), (2.2) that

$$\int_0^{\omega} \{(1-\mu)\, c_i\, x + \mu f(x)\}\, dt = 0 \qquad (i = 1, 2). \tag{4.6}$$

Since $c_i > 0$ $(i = 1, 2)$ and $f$ is subject to (1.5) it is clear from (4.6) with $0 < \mu < 1$ that

$$|x(\tau_0)| < 1 \qquad \text{for some } \tau_0 \text{ such that } \quad 0 \leq \tau_0 \leq \omega\,. \tag{4.7}$$

Now the coefficient $a_0$ in (4.2) is given by

$$\begin{aligned} a_0 &= \omega^{-1} \int_0^{\omega} x(t)\, dt \\ &= \omega^{-1} \int_{\tau_0}^{\tau_0+\omega} x(t)\, dt\,, \end{aligned}$$

since $x(t)$ is $\omega$-periodic in $t$. Now

$$\begin{aligned} \int_{\tau_0}^{\tau_0+\omega} x(t)\, dt &= [t x(t)]_{\tau_0}^{\tau_0+\omega} - \int_{\tau_0}^{\tau_0+\omega} t\dot{x}(t)\, dt\,. \\ &= \omega x(\tau_0) - \int_{\tau_0}^{\tau_0+\omega} t\dot{x}(t)\, dt\,, \end{aligned}$$

so that, by (4.7),

$$|a_0| < 1 + \omega^{-1} \int_{\tau_0}^{\tau_0+\omega} t\, |\dot{x}(t)|\, dt$$

and therefore, since $0 \leq \tau_0 \leq \omega$,

$$\begin{aligned} |a_0| &\leq 1 + D \int_{\tau_0}^{\tau_0+\omega} |\dot{x}(t)|\, dt \\ &\leq 1 + D \left( \int_{\tau_0}^{\tau_0+\omega} \dot{x}^2\, dt \right)^{\frac{1}{2}} \end{aligned}$$

by Schwarz's inequality. Hence

$$a_0^2 \leq D_2 \left( 1 + \int_{\tau_0}^{\tau_0+\omega} \dot{x}^2\, dt \right) \tag{4.8}$$

for sufficiently large $D_2$. As for the term under the summation sign in (4.5) it is clear by comparison with (4.3) that

$$\sum_{r=1}^{\infty} (a_r^2 + b_r^2) \leq D \int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt . \tag{4.9}$$

The result (4.4) now follows on combining (4.8) and (4.9) with (4.5).

5. PROOF OF THEOREM 1. Let now $x = x(t)$ be any $\omega$-periodic solution of (2.1) with $0 < \mu < 1$ and $\phi$ subject to (1.3).

Define $I_0 \geq 0$, $I_1 \geq 0$, $I_2 \geq 0$ by:

$$I_0^2 = \int_0^{\omega} x^2 \, dt \quad , \quad I_1^2 = \int_0^{\omega} \dot{x}^2 \, dt \quad , \quad I_2^2 = \int_0^{\omega} \ddot{x}^2 \, dt .$$

Since

$$\int \dot{x}\dddot{x} \, dt = \dot{x}\ddot{x} - \int \ddot{x}^2 \, dt \qquad \text{and} \qquad \int \phi(x)\, \dot{x}^2 \, dt = \dot{x}\Phi(x) - \int \ddot{x}\Phi(x) \, dt$$

we have, on multiplying (2.1) by $\dot{x}$ and integrating, that

$$I_2^2 + \mu \int_0^{\omega} \Phi(x)\, \ddot{x} \, dt = -\mu \int_0^{\omega} \dot{x} p(t) \, dt ,$$

so that, by (1.3) and since $0 < \mu < 1$,

$$\begin{aligned} I_2^2 &\leq B_1 \int_0^{\omega} |x|\,|\ddot{x}| \, dt + \left\{ B_2 \int_0^{\omega} |\ddot{x}| \, dt + A_3 \int_0^{\omega} |\dot{x}| \, dt \right\} \\ &\leq B_1 I_0 I_2 + \omega^{\frac{1}{2}} (B_2 I_2 + A_3 I_1) , \end{aligned} \tag{5.1}$$

by Schwarz's inequality. But, by (4.4),

$$\begin{aligned} I_0 &\leq D_0 + D_1 I_1 \\ &\leq D_3 (1 + I_2) , \end{aligned} \tag{5.2}$$

by (4.1), for sufficiently large $D_3$. Thus (5.1) also implies that

$$I_2^2 \leq D_3 B_1 I_2^2 + (B_1 D_3 + D) I_2$$

by (4.1); and hence if $B_1$ is fixed, as we assume henceforth, such that

$$B_1 \leq \tfrac{1}{2} D_3^{-1}, \tag{5.3}$$

then

$$I_2^2 \leq D I_2 ,$$

from which it follows at once that

$$I_2^2 \leq D_4 , \tag{5.4}$$

and then also, by (4.1), that

$$I_1^2 \leq D_5 . \tag{5.5}$$

Now a combination of (4.7) with the identity:

$$x(t) \equiv x(\tau_0) + \int_{\tau_0}^{t} \dot{x}(s)\, ds$$

shows that

$$\max_{0 \leq t \leq \omega} |x(t)| < 1 + \int_{\tau_0}^{\tau_0+\omega} |\dot{x}(s)|\, ds$$

$$\leq 1 + \omega^{\frac{1}{2}} \left( \int_{\tau_0}^{\tau_0+\omega} \dot{x}^2(s)\, ds \right)^{\frac{1}{2}}$$

by Schwarz's inequality. Hence, by (5.5),

$$|x(t)| \leq D_6 \equiv 1 + \omega^{\frac{1}{2}} D_5^{\frac{1}{2}} \qquad (0 \leq t \leq \omega) . \tag{5.6}$$

Next, since $x(0) = x(\omega)$ it is clear that $\dot{x}(\tau_1) = 0$ for some $\tau_1 \in [0 , \omega]$. Thus we have, as a result of the identity:

$$\dot{x}(t) = \dot{x}(\tau_1) + \int_{\tau_1}^{t} \ddot{x}(s)\, ds ,$$

that

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq \int_{\tau_1}^{\tau_1+\omega} |\ddot{x}(s)|\, ds$$

$$\leq \omega^{\frac{1}{2}} \left( \int_{\tau_1}^{\tau_1+\omega} \ddot{x}^2(s)\, ds \right)^{\frac{1}{2}},$$

by Schwarz's inequality, and therefore, by (5.4), that

$$|\dot{x}(t)| \leq D_7 \equiv \omega^{\frac{1}{2}} D_4^{\frac{1}{2}} \qquad (0 \leq t \leq \omega) . \tag{5.7}$$

It remains now to establish the last estimate in (2.5). For this let us note from (2.1) that $\dddot{x} = Q$, where by virtue of (5.6) and (5.7) and the boundedness of $p$ the function Q satisfies

$$|Q| \leq D_8 (|\ddot{x}| + 1) .$$

Thus if we multiply both sides of (2.1) by $\dddot{x}$ and integrate we shall obtain that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt \leq D_8 \int_{\tau}^{\tau+\omega} |\ddot{x}|\,|\dddot{x}|\,dt + D_8 \int_{\tau}^{\tau+\omega} |\dddot{x}|\,dt$$

$$\leq D\left(\int_{\tau}^{\tau+\omega} \ddot{x}^2\,dt\right)^{\frac{1}{2}}\left(\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt\right)^{\frac{1}{2}} + D\left(\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt\right)^{\frac{1}{2}}$$

by Schwarz's inequality. Hence, by (5.4),

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt \leq D\left(\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt\right)^{\frac{1}{2}}$$

which in turn implies that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2\,dt \leq D_9\,. \tag{5.8}$$

Now, since $\dot{x}(0) = \dot{x}(\omega)$ it follows that $\ddot{x}(\tau_2) = 0$ for some $\tau_2 \in [0, \omega]$. Therefore we have, from the identity:

$$\ddot{x}(t) = \ddot{x}(\tau_2) + \int_{\tau_2}^{t} \dddot{x}(s)\,ds\,,$$

that

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \omega^{\frac{1}{2}}\left(\int_{\tau_2}^{\tau_2+\omega} \dddot{x}^2(s)\,ds\right)^{\frac{1}{2}} \leq D$$

by (5.8).

This completes the verification of (2.5) for all $\omega$-periodic solutions of (2.1) with $0 < \mu < 1$ and Theorem 1 now follows with $\varepsilon_0 = \frac{1}{2} D_3^{-1}$ (See (5.3)).

6. PROOF OF THEOREM 2. We deal first with the case $\psi$ subject to (1.7). Let then $x = x(t)$ be any $\omega$-periodic solution of (2.2) with $0 < \mu < 1$. The whole substance of our proof, as pointed out in § 2 will be to establish (2.5) for $x(t)$. With the groundwork laid out in § 4 the pattern for the proof of (2.5) here is almost as in § 5 and we shall therefore skip any inessential details.

Indeed the main difference between our procedure here and the procedure in § 5 is in the method for estimating $\int_0^{\omega} \dot{x}^2\,dt$. This time it is convenient to multiply our parameter-dependent equation (2.2) by $x$ (not by $\dot{x}$ as in § 5) and then integrate. Since

$$\int x\dddot{x}\,\mathrm{d}t = x\ddot{x} - \tfrac{1}{2}\dot{x}^2 \quad , \quad \int x\ddot{x}\,\mathrm{d}t = x\dot{x} - \int \dot{x}^2\,\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^x \xi\phi(\xi)\,\mathrm{d}\xi = x\phi(x)\,\dot{x} \quad , \quad \int x\psi(\dot{x})\,\ddot{x}\,\mathrm{d}t = x\Psi(\dot{x}) - \int \dot{x}\Psi(\dot{x})\,\mathrm{d}t$$

where $\Psi(y) \equiv \int_0^y \psi(\eta)\,\mathrm{d}\eta$, and $x$ is $\omega$-periodic, the integration leads at once to the result:

$$(6.1) \quad (1-\mu)\,\alpha\int_0^\omega \dot{x}^2\,\mathrm{d}t + \mu\int_0^\omega \dot{x}\Psi(\dot{x})\,\mathrm{d}t = \int_0^\omega \{(1-\mu)\,c_2\,x^2 + \mu x f(x) - \mu x p\}\,\mathrm{d}t\,.$$

By (1.7) $\psi \geq \alpha$ and therefore also $y\Psi(y) \geq \alpha y^2$ for all $y$.

Thus the inequality (6.1), if (1.2) holds, implies that

$$(6.2) \quad \int_0^\omega \dot{x}^2\,\mathrm{d}t \leq \alpha^{-1}(c_2 + \mathrm{A}_1)\int_0^\omega x^2\,\mathrm{d}t + \mathrm{D}\int_0^\omega |x|\,\mathrm{d}t$$

$$\leq \alpha^{-1}(c_2 + \mathrm{A}_1)\int_0^\omega x^2\,\mathrm{d}t + \mathrm{D}\left(\int_0^\omega x^2\,\mathrm{d}t\right)^{\frac{1}{2}},$$

by Schwarz's inequality. By (4.4) and (5.2) this implies in turn that

$$(6.3) \quad \int_0^\omega \dot{x}^2\,\mathrm{d}t \leq \alpha^{-1}(c_2 + \mathrm{A}_1)\,\mathrm{D}_1^2\int_0^\omega \dot{x}^2\,\mathrm{d}t + \mathrm{D}\left\{\left(\int_0^\omega \dot{x}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \mathrm{A}_1 + 1\right\}.$$

Hence if for example $c_2$ and $\mathrm{A}_1$ are fixed, as we assume henceforth, such that

$$(6.4) \quad 0 < c_2 < \frac{1}{4}\,\alpha\mathrm{D}_1^{-2} \quad , \quad \mathrm{A}_1 \leq \frac{1}{4}\,\alpha\mathrm{D}_1^{-2}$$

then we have from (6.3) that

$$\int_0^\omega \dot{x}^2\,\mathrm{d}t \leq \mathrm{D}\left\{\left(\int_0^\omega \dot{x}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + 1\right\}$$

which, in turn leads to (5.5) and therefore to (5.6) as in § 5.

It remains now to obtain the estimates for $|\dot{x}(t)|$ and $|\ddot{x}(t)|$ in (2.5).

The estimate for $|\dot{x}(t)|$ requires (5.4), just as in § 5, and to establish this we note that (2.2) implies that

$$\dddot{x} + \{(1 - \mu)\,\alpha + \mu\psi(\dot{x})\}\,\ddot{x} = \mathrm{R} \tag{6.5}$$

where, because of the boundedness, just established, of $|x(t)|$ by a D, the function R satisfies

$$|\mathrm{R}| \leq \mathrm{D}\,(|\dot{x}| + 1)\,.$$

Thus if we multiply both sides of (6.5) by $\ddot{x}$ and integrate we shall have, since $x$ is $\omega$-periodic and $(1 - \mu)\,\alpha + \mu\psi \geq \alpha$, that

$$\begin{aligned}
\alpha \int_0^\omega \ddot{x}^2\, dt &\leq \mathrm{D}\left(\int_0^\omega |\dot{x}|\,|\ddot{x}|\, dt + \int_0^\omega |\ddot{x}|\, dt\right) \\
&\leq \left\{\left(\int_0^\omega \dot{x}^2\, dt\right)^{\frac{1}{2}}\left(\int_0^\omega \ddot{x}^2\, dt\right)^{\frac{1}{2}} + \left(\int_0^\omega \ddot{x}^2\, dt\right)^{\frac{1}{2}}\right\} \\
&\leq \mathrm{D}\left(\int_0^\omega \ddot{x}^2\, dt\right)^{\frac{1}{2}}
\end{aligned}$$

by (5.6) which has just been established for $\omega$-periodic solutions of (2.2). Hence

$$\int_0^\omega \ddot{x}^2\, dt \leq \mathrm{D}$$

as before and the estimate (5.7) then follows as in § 5 for our solution $x$ of (2.2).

With the boundedness (each by a D) of $|x(t)|$ and $|\dot{x}(t)|$ established, the estimate (5.8.) can now follow, for our solution of (2.2) exactly as in § 5, and so also the boundedness of $|\ddot{x}(t)|$ by a D for arbitrary $t \in [0, \omega]$. This concludes the verification of Theorem 2 with $\varepsilon_1 = \frac{1}{4}\alpha \mathrm{D}_1^{-2}$ (see (6.4)) when $\psi$ is subject to (1.7).

To tackle the case $\psi$ subject to (1.8) we had pointed out in § 2 that we should deal with the equation (2.2) with $\alpha$ replaced by $(-\beta)$. The effect of the replacement on the estimate for $\int_0^\omega \dot{x}^2\, dt$ is merely to replace $\alpha^{-1}$ in (6.2) by $\beta^{-1}$, as is easily seen by multiplying both sides of (6.1) by $(-1)$ and noting that $-\mu y\,\Psi^{*}(y) \geq \mu\beta y^2$ so that then

$$(1-\mu)\,\beta\int_0^{\omega}\dot{x}^2\,\mathrm{d}t-\mu\int_0^{\omega}\dot{x}\Psi^{*}(\dot{x})\,\mathrm{d}t\geq\beta\int_0^{\omega}\dot{x}^2\,\mathrm{d}t\,.$$

Thus the estimate (6.3) comes through here with $\beta$ in place of $\alpha$ and the rest of the proof when $\psi$ is subject to (1.8) can now follow from that point exactly as before.

This completes our proof of Theorem 2.

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