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Generic functions on a stratified space

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Generic functions on a stratified space* (*). Nota (**) di ROBERTO PIGNONI, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Enunciamo teoremi di densità e stabilità per le funzioni di Morse su uno spazio stratificato immerso in $\mathbf{R}^n$. Le dimostrazioni saranno presto pubblicate.

The methods of Morse theory can be extended to a wider category of spaces than differentiable manifolds, for which the classical theory was conceived.

In [2] Lazzeri has shown how these methods can be generalized to the case of spaces with isolated singularities, establishing in this way results on the homotopy type of Stein spaces and on the homology of singular projective algebraic varieties.

Benedetti has given in [1] a definition of a "Morse function" in the case of any stratified space, generalizing the notion already given in [2] in the case of isolated singularities. We proceed to illustrate this definition for stratified subsets of $\mathbf{R}^n$.

A subset $X \subset \mathbf{R}^n$ is a *stratified space* if a well determined stratification is given on it. For any stratum S of the stratification of X one defines the limit planes in the points of ∂S.

We say that $f: X \to \mathbf{R}$ is a function of class C^k, $0 \leq k \leq \infty$, if it is the restriction to X of a C^k function defined over the whole of $\mathbf{R}^n$. By $C^k(X, \mathbf{R})$ we shall mean the space of k-differentiable functions on X, endowed with the Whitney topology, and by $C_0^k(X, \mathbf{R})$ the subspace of proper functions.

(*) Lavoro eseguito nell'ambito del GNSAGA del C.N.R.
(**) Pervenuta all'Accademia il 12 settembre 1978.

Let X be a stratified space. A function $f \in C^k(X, \mathbf{R})$, $2 \leq k \leq \infty$, is said to be a *Morse function* on X if

1) for any stratum U of X, $\dim U > 0$, $f_{|U}$ has only nondegenerate critical points;
2) for any stratum U of X and any limit plane T of U in any point $p \in \partial U$, one has that $(Df)_p$ is not null on T.

We emphasize that the notion of Morse function is not defined with respect to the set X, but depends strictly on the particular stratification that has been chosen for X.

We say that $f \in C_0^k(X, \mathbf{R})$ is *stable* (i.e., topologically stable) when there is an open neighbourhood N of f in $C_0^k(X, \mathbf{R})$ such that $\forall g \in N$ one can find homeomorphisms $h_1 : X \to X$ and $h_0 : \mathbf{R} \to \mathbf{R}$ for which $g \circ h_1 = h_0 \circ f$.

In order to extend the methods of Morse theory to the case of more general spaces than the ones treated in [2], we need to prove that, when X is a stratified subset of $\mathbf{R}^n$ with good enough properties, the Morse functions with respect to its stratification are an open dense set in $C_0^k(X, \mathbf{R})$, $3 \leq k \leq \infty$, and are stable.

We have established two theorems of this kind:

THEOREM 1. *Let* X *be a stratified space in* $\mathbf{R}^n$ *with strata which are analytic sub-manifolds and semi-analytic subsets in* $\mathbf{R}^n$. *Then, Morse functions over this stratification are dense in* $C^k(X, \mathbf{R})$, $2 \leq k \leq \infty$.

An obvious corollary of Theorem 1 is that Morse functions with distinct critical values (i.e., $f(p_1) \neq f(p_2)$ if $p_1 \neq p_2$ are critical points in some strata, or are 0-dimensional strata) are dense in $C^k(X, \mathbf{R})$ and $C_0^k(X, \mathbf{R})$, $2 \leq k \leq \infty$.

THEOREM 2. *Morse functions with distinct critical values are stable in* $C_0^k(X, \mathbf{R})$, $3 \leq k \leq \infty$, *whenever* X *is a stratified space for which hold Whitney's conditions a and b.*

A consequence of these two theorems is the following fact.

Let X be a closed semi-analytic subset of $\mathbf{R}^n$.

Lojasiewicz [3] has shown that X has a stratification which has all the properties assumed in the hypothesis of Theorem 1 and Theorem 2, and is canonically associated to X since it is "minimal" among all possible stratifications with the same properties. Then, Morse functions over this stratification of X are a generic family of stable functions in $C_0^k(X, \mathbf{R})$, $3 \leq k \leq \infty$.

REFERENCES

[1] R. BENEDETTI (1977) - *Density of Morse functions on a complex space*, «Math. Ann.», *229*, 135-139.

[2] F. LAZZERI (1973) - *Morse theory on singular spaces* «Astérisque», 7 et *8*.

[3] S. LOJASIEWICZ (1975) - *Ensembles semi-analytiques*, Preprint «I.H.E.S.».