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A structural property of prime ideals in a topological noetherian algebra with an application to complex analysis

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A structural property of prime ideals in a topological noetherian algebra with an application to complex analysis.* Nota di EDOARDO BALLICO e ARTURO V. FERREIRA, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si stabilisce una formula d'intersezione per un ideale primo di un'algebra topologica neotheriana e se ne ricava un teorema degli zeri per certi compatti di uno spazio analitico complesso.

1. Let A be a complex (unitary) noetherian algebra with a complete barrelled Hausdorff topology which can be defined by a system of algebra semi-norms. We suppose A possesses the open mapping property in the sense of [2]; we have

THEOREM. *Every prime ideal p in* A *is the intersection of those prime ideals $q \supset p$ for which the Krull dimension of* A/q *is* ≤ 1.

The basic information about noetherian topological algebras used in the proof given in section 2, namely the fact that ideals are closed, is to be found in [2]. For convenience, the statement of the Corollary which contains an application to complex analysis will be preceded by some preliminary considerations.

Let X be a complex analytic space holomorphically separated (which can be supposed reduced without loss of generality) and K a compact

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subset of X. The theorem clearly applies to the algebra O (K) of holomorphic sections over K endowed with its usual Silva inductive limit topology, whenever O (K) is noetherian. The compact K is said to be *holomorphically convex* if every character of O (K) is defined by the evaluation at some point of K or, equivalently, every maximal ideal of O (K) is the ideal of the germs in O (K) which vanish at a point in K.

A germ of analytic set on K can be associated in the usual way to each ideal I of O (K) and will be denoted by loc(I). Conversely, each germ of analytic set S on K defines the ideal idl(S) constituted by the elements in O (K) which vanish on S in an obvious sense. We will say that the Nullstellsatz holds for O (K) if for each ideal I in O (K), idl(loc(I)) is just the nil–radical of I.

The validity of such a zero's theorem is obviously related to finiteness properties of the ring O (K), and here the natural assumption is that O (K) is a noetherian complex algebra.

COROLLARY. *Let* O (K) *be noetherian and suppose the holomorphically convex compact* K *has a fundamental system of open neighbourhoods which have envelop of holomorphy, then the Nullstellensatz holds for* O (K). *In particular, when* X *is a Stein manifold the Nullstellensatz holds for* O (K) *whenever* K *is holomorphically convex and* O (K) *noetherian.*

Remarks 1. Nullstellensatz property holds surely in much more general situations; we hope to return later on the subject after the appropriate tools will be developed in a paper of the series begun by [2].

2. It is probably not true that the noetherianity of O (K), K holomorphically convex compact, implies K is a Stein compact in the sense each neighbourhood contains an open Stein neighbourhood. In [1], J.-E. Björk gives an example of a holomorphically convex compact K in C^2 which is not Stein; in this example K has an infinity of connected components and so O (K) cannot be noetherian by theorem 2.1 in [2].

2. Proof of the theorem. In a local noetherian algebra R every prime ideal is the intersection of the prime ideals q containing it such that $\dim (R/q) \leq 1$, where the symbol dim stands for Krull dimension, cfr. Langmann [3]. Consider now our algebra A and a prime ideal p of A; p being closed, we can without loss of generality suppose moreover A is a domain of integrity and $p = 0$.

Fix a maximal ideal M_0 in A and take the localized A_{M_0} of A at M_0. A_{M_0} is a local noetherian integral domain so that we must have in A_{M_0}, $0 = \bigcap_{Q \in S} Q$ where S denotes the set of prime ideals Q with $\dim (A_{M_0}/Q) \leq 1$. For every $Q \in S$, $q_Q = Q \cap A$ is a prime ideal in A for which the chain $M_0 \supset q_Q$ cannot be refined as a chain of prime ideals, and also we will clearly have $0 = \bigcap_{Q \in S} q_Q$. That being, the proof of the theorem consists just in the verification that *each* q_Q, $Q \in S$, *is the intersection of a family of prime ideals* q' *in* A *with* $\dim(A/q') \leq 1$.

We shall proceed by absurd. Take Q in S and suppose our claim is false for that Q. Let Φ be the subset of the maximal ideal space $\Sigma(A)$ of A constituted by the maximal ideals M such that $M \supset q_Q$ cannot be refined as a chain of prime ideals. We have $\Phi \neq \varnothing$ because $M_0 \in \Phi$. Put $B = A/q_Q$ and identify in the natural way Φ to a subset of $\Sigma(B)$; by hypothesis $\Psi = \Sigma(B) \setminus \Phi$ is also non-void.

Now, for every $M' \in \Psi$ choose a prime ideal $q(M') \subset M'$ with $\dim(B/q(M')) \leq 1$ and define the ideals in B: $I' = \bigcap_{M' \in \Psi} q(M')$, $I = \bigcap_{M \in \Phi} M$; $I, I' \neq 0$ because the stated claim fails for Q. We must have a Lasker-Noether decomposition $I = M_1 \cap \cdots \cap M_s$ for a finite family of elements of Φ and $I' = p_1 \cap \cdots \cap p_r$ for a finite family of prime ideals $\neq 0$ each of which is contained in some $q(M')$, $M' \in \Psi$.

There results that: (i) Φ is a finite closed set in $\Sigma(B)$ for the Zariski topology, and (ii) Ψ is also a closed subset of $\Sigma(B)$ for the Zariski topology. The Zariski topology is coarser than the usual Gel'fand topology on $\Sigma(B)$ and therefore, $\Sigma(B)$ will be disconnected, which implies the existence in B of an idempotent $e \neq 0, 1$ by Šilov's idempotent theorem. We thus obtain the contradictory relation $e(e-1) = 0$ in the integral domain B.

It follows that our claim must be true for all $Q \in S$.

Proof of the corollary. By using the theorem we can restrict ourselves to establish that for every prime ideal q in $O(K)$ with $\dim(O(K)/q) \leq 1$, we have $\text{idl}(\text{loc}(q)) = q$. This assertion is obviously true if $\dim(O(K)/q) = 0$ because, K being holomorphically convex, we cannot have $\text{loc}(q) = \varnothing$ for a maximal ideal q. Now, consider a prime ideal q with $\dim(O(K)/q) = 1$ and suppose $\text{idl}(\text{loc}(q)) \supset q$. To prove the corollary it is enough to derive a contradition from this hypothesis.

First we observe that we necessarily have a Lasker-Noether decomposition $\text{idl}(\text{loc}(q)) = M_0 \cap \cdots \cap M_s$ where $M_0, \cdots, M_s$ is a finite family of maximal ideals in $O(K)$. This implies $\Sigma(O(K)/q) = \{M_0, \cdots, M_s\}$ so that we must actually have $s = 0$ by Šilov's idempotent theorem, because $O(K)/q$ is a domain of integrity. We will indicate by x_0 the point of K which corresponds to the maximal ideal M_0. If $f_1, \cdots, f_k$ is a finite system of generators for q over $O(K)$, there must be some neighbourhood V_0 of K such that f_i, $i = 1, \cdots, k$, are holomorphic in V_0 and the set of common zeros of $f_1, \cdots, f_k$ reduces to the point x_0.

Take an arbitrary open neighbourhood $V \subset V_0$ of K which possesses envelope of holomorphy $\tilde{V}$. We shall identify the structure space $\Sigma(O(V))$ of the topological algebra $O(V)$ of holomorphic sections to $\tilde{V}$, and the Gel'fand transform of each $f \in O(V)$ with its analytic continuation $\tilde{f}$.

We must have in $\tilde{V}$, $\tilde{f}_1^{-1}(0) \cdots \tilde{f}_k^{-1}(0) = \{x_0\} \cup \gamma$, where γ is a closed subset which does not contain x_0.

Denote by I the ideal generated by $\tilde{f}_1, \cdots, \tilde{f}_k$ in $O(\tilde{V})$ and by q_V the prime ideal constituted by the $f \in O(\tilde{V})$ such that the germ of f on K belongs to q. I and q_V are clearly closed ideals.

The structure space $\Sigma(O(\tilde{V})/I)$ can be identified to $\{x_0\} \cup \gamma$ and therefore, we can apply Šilov's idempotent theorem to conclude that exists e in $O(\tilde{V})$ such that $e(x_0) = 0$, $e(\gamma) = \{1\}$. Now, the class $\bar{e}$ determined by e in the integral domain $O(\tilde{V})/q_V$ verifies the relation $\bar{e}(1 - \bar{e}) = 0$ from which we can obviously deduce that $e \in q_V$.

Put $\tilde{f}_0 = e$ and denote by I_0 the ideal generated by $\tilde{f}_0, \cdots, \tilde{f}_k$ in $O(V)$. The finitely generated ideal I_0 is contained in the unique maximal ideal M_{0V} determined by x_0 and also in q_V; as $\tilde{V}$ is a Stein space, we can conclude by a straighforward cohomological argument that $M_{0V} = q_V$. Hence, $M_0 = q$ because V is arbitrary, which is in contradiction with the hypothesis $\dim(O(K)/q) = 1$.

The last assertion in the statement of the Corollary results from [4].

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