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The projective geometry of the Weyl spinor

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Geometria differenziale. — *The projective geometry of the Weyl spinor.* Nota di JOSEPH ZUND, presentata (*) dal Socio G. SANSONE.

SUMMARY. — See the Introduction.

INTRODUCTION

In this note the eigenbispinors of the Weyl spinor are studied and interpreted using projective geometry. It is shown that the author's strong form of the Bel-Petrov classification can be represented in terms of a set of lines and a conic in the complex projective plane.

1. THE BIVECTOR HEXAD AND BISPINORS

In his study of the complex null tetrad $\underset{\sim}{Z}^{\mu} = \{l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}\}$ Sachs has shown how to construct a bivector hexad consisting of three self-dual and three anti-self-dual complex bivectors. These bivectors are defined by

$$\begin{aligned} U^{\mu\nu} &\equiv 2\, n^{[\mu} m^{\nu]} \\ V^{\mu\nu} &\equiv 2\, l^{[\mu} \bar{m}^{\nu]} \\ W^{\mu\nu} &\equiv 2\, l^{[\mu} n^{\nu]} + 2\, m^{[\mu} \bar{m}^{\nu]}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \bar{U}^{\mu\nu} &\equiv 2\, n^{[\mu} \bar{m}^{\nu]} \\ \bar{V}^{\mu\nu} &\equiv 2\, l^{[\mu} m^{\nu]} \\ \bar{W}^{\mu\nu} &\equiv 2\, l^{[\mu} n^{\nu]} - 2\, m^{[\mu} \bar{m}^{\nu]} \end{aligned} \tag{1.2}$$

respectively. In [1] it was shown that the complex null tetrad $\underset{\sim}{Z}^{\mu}$, corresponds to the spinor zweibein $\underset{\sim}{Z}^{A} \equiv \{\lambda^{A}, \mu^{A}\}$ according to the scheme

$$\begin{aligned} l^{\mu} &\leftrightarrow \lambda^{A} \bar{\lambda}^{\dot{X}} & \qquad m^{\mu} &\leftrightarrow \lambda^{A} \bar{\mu}^{\dot{X}} \\ n^{\mu} &\leftrightarrow \mu^{A} \bar{\mu}^{\dot{X}}, & \qquad \bar{m}^{\mu} &\leftrightarrow \mu^{A} \bar{\lambda}^{\dot{X}} \end{aligned} \tag{1.3}$$

with

$$\varepsilon_{AB} = 2\, \lambda_{[A} \mu_{B]} \tag{1.4}$$

and $\lambda_{A} \mu^{A} = 1$.

(*) Nella seduta del 13 gennaio 1979.

(1) Throughout this note we will employ the notation and terminology employed in in [1] and [2]. The latter paper contains a complete bibliography.

We now determine the spinor version of the bivector hexad

$$\underset{\sim}{Z}^{\mu\nu} = \{U^{\mu\nu}, V^{\mu\nu}, W^{\mu\nu}, \bar{U}^{\mu\nu}, \bar{V}^{\mu\nu}, \overline{W}^{\mu\nu}\} .$$

By using (1.3) and (1.4), the spinors corresponding to the self-dual bivectors are

$$\begin{aligned} U^{A\dot{X}B\dot{Y}} &= \bar{u}^{\dot{X}\dot{Y}}\, \varepsilon^{AB} \\ V^{A\dot{X}B\dot{Y}} &= \bar{v}^{\dot{X}\dot{Y}}\, \varepsilon^{AB} \\ W^{A\dot{X}B\dot{Y}} &= \bar{w}^{\dot{X}\dot{Y}}\, \varepsilon^{AB} \end{aligned} \tag{1.5}$$

where we have defined

$$u^{AB} \equiv -\mu^A \mu^B,\ v^{AB} \equiv \lambda^A \lambda^B,\ w^{AB} \equiv 2\lambda^{(A} \mu^{B)} . \tag{1.6}$$

The spinors corresponding to (1.2) are merely the complex conjugates of those given in (1.5). Thus the set of symmetric spinors given in (1.6) constitute a basis of the space of undotted second order symmetric spinors. It will be convenient to call a second order symmetric spinor a *bispinor*. Hence we may say that the bivector hexad $\underset{\sim}{Z}^{\mu\nu}$ defines a pair of bispinor triads $\underset{\sim}{Z}^{AB} \equiv (u^{AB}, v^{AB}, w^{AB}\}$ and $\underset{\sim}{\bar{Z}}^{\dot{X}\dot{Y}} \equiv \{\bar{u}^{\dot{X}\dot{Y}}, \bar{v}^{\dot{X}\dot{Y}}, w^{\dot{X}\dot{Y}}\}$. These bispinors satisfy the properties

$$\begin{aligned} &u_{AB}\, u^{AB} = v_{AB}\, v^{AB} = u_{AB}\, w^{AB} = v_{AB}\, w^{AB} = 0 \\ &u_{AB}\, v^{AB} = -1\ ,\ w_{AB}\, w^{AB} = 2 \end{aligned} \tag{1.7}$$

which are analogous to the orthogonality conditions of the tetrad $\underset{\sim}{Z}^{\mu}$.

2. THE BEL–PETROV CLASSIFICATION AND EIGENBISPINORS

In terms of the spinor zweibein $\underset{\sim}{Z}^A \equiv \{\lambda^A, \mu^A\}$ the Weyl spinor may be expressed in the form

$$\begin{aligned} \psi_{ABCD} = {} & \Psi_0\, \mu_A \mu_B \mu_C \mu_D - 4\, \Psi_1\, \mu_{(A} \mu_B \mu_C \lambda_{D)} + 6\, \Psi_2\, \mu_{(A} \mu_B \lambda_C \lambda_{D)} \\ & - 4\, \Psi_3\, \mu_{(A} \lambda_B \lambda_C \lambda_{D)} + \Psi_4\, \lambda_A \lambda_B \lambda_C \lambda_D . \end{aligned} \tag{2.1}$$

The complex coefficients $\Psi_0, \Psi_1, \cdots, \Psi_4$ are called the *Weyl coefficients* and they determine the various Bel-Petrov types. This classification has been given in several forms, and the one presented below (which we call the *strong form* of the Bel–Petrov classification) is based on a detailed investigation of Cayley's theorem [1], [2].

The strong form of the Bel-Petrov classification, based on the principal null direction l^{μ}, is as follows:

Type I: $\Psi_0 = \Psi_4 = 0,\ \Psi_1 \Psi_2 \Psi_2 \neq 0$ with $3\Psi_2^2 - 4\Psi_1 \Psi_3 \neq 0$ and $\Psi_2^2 - 2\Psi_1 \Psi_3 \neq 0$.

Type II: $\Psi_0 = \Psi_1 = 0, \Psi_2 \neq 0$ with either $\Psi_3 = 0$, $\Psi_4 \neq 0$ or $\Psi_3 \neq 0$, $\Psi_4 = 0$. (These will be called Types II a and II b respectively)

Type D: $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\,, \Psi_2 \neq 0\,.$

Type III: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0\,, \Psi_3 \neq 0\,.$

Type N: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,, \Psi_4 \neq 0\,.$

Type 0: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ (2).

In terms of the covariant form of the bispinor triad $\underset{\sim}{Z}^{AB}$, the Weyl spinor (2.1) may be re-expressed in the bispinor form

$$
\begin{aligned}
\psi_{ABCD} = {} & \Psi_0\, u_{AB}\, u_{CD} + \Psi_1 \left(u_{AB}\, w_{CD} + w_{AB}\, u_{CD}\right) \\
& + \Psi_2 \left(- u_{AB}\, v_{CD} - v_{AB}\, u_{CD} + w_{AB}\, w_{CD}\right) \\
& - \Psi_3 \left(v_{AB}\, w_{CD} + w_{AB}\, v_{CD}\right) + \psi_4\, v_{AB}\, v_{CD}\,.
\end{aligned} \tag{2.2}
$$

We now seek eigenbispinors of $\psi^{AB}{}_{CD}$, viz bispinors k^{AB} such that

$$
\psi^{AB}{}_{CD}\, k^{CD} = \Lambda k^{AB} \tag{2.3}
$$

where Λ is a scalar which may or may not vanish. Thus by use of (1.7) and (2.2) it is easy to verify that

$$
\begin{aligned}
\psi^{AB}{}_{CD}\, u^{CD} &= \Psi_2\, u^{AB} - \Psi_4\, v^{AB} \\
\psi^{AB}{}_{CD}\, v^{CD} &= - \Psi_0\, u^{AB} + \Psi_2\, v^{AB} - \Psi_1\, w^{AB} \\
\psi^{AB}{}_{CD}\, w^{CD} &= - 2\, \Psi_1\, u^{AB} + 2\, \Psi_3\, v^{AB} - 2\, \Psi_2\, w^{AB}.
\end{aligned} \tag{2.4}
$$

Examination of these equations together with the conditions for the various Bel-Petrov types yields the following classification in terms of eigenbispinors of the Weyl spinor.

THEOREM.

The strong form of the Bel-Petrov classification defines the following eigenbispinors and eigenvalues:

Type I: *Eigenbispinor*: u^{AB}; *Non-zero eigenvalue*: Ψ_2.

Type IIa: *Eigenbispinors*: v^{AB}, w^{AB}; *Non-zero eigenvalues*: $\Psi_2, -2\Psi_2$.

Type IIb: *Eigenbispinors*: w^{AB}, v^{AB}; *Non-zero eigenvalues*: Ψ_2, Ψ_2.

Type D: *Eigenbispinors*: u^{AB}, v^{AB}, w^{AB}; *Non-zero eigenvalues*: $\Psi_2, \Psi_2, -2\,\Psi_2$.

(2) Note that Type 0 has $\psi_{ABCD} = 0$, and is included here only for purposes of formal completeness. We will always assume that our spacetime is not of Type 0.

Type III: *Eigenbispinors*: u^{AB}, v^{AB}; *Zero eigenvalues.*

Type N: *Eigenbispinors*: v^{AB}, w^{AB}; *Zero eigenvalues.*

Penrose has illustrated the Bel-Petrov classification in terms of principal null directions bu the following diagram (Fig. 1) in which the arrows denote the repeated coincidence of the principal null directions.

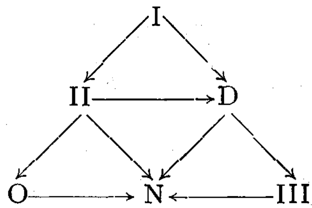

Fig. 1.

The Bel–Petrov classification in terms of eigenbispinors given in the previous theorem may also be summarized in diagram form. This classification is illustrated in Fig. 2 where nonvertical arrows indicate the *decrease* in the number of eigenbispinors (which preserves the eigenvalues involved) and vertical arrows indicate a specialization (which preserves the eigenbispinors) requiring that the non-zero eigenvalues vanish.

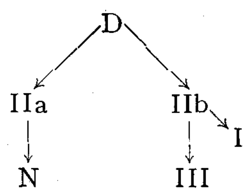

Fig. 2.

3. THE PROJECTIVE GEOMETRY OF THE BISPINOR CLASSIFICATION

Let $P_2(\mathbf{C})$ denote the complex projective plane. There are two ways of geometrically visualizing the classification indicated in the theorem of Section 2. The first approach is derived by using the Plücker spinor

$$p^{A\dot{X}B\dot{Y}} = \rho^{AB}\varepsilon^{\dot{X}\dot{Y}} + \bar{\rho}^{\dot{X}\dot{Y}}\varepsilon^{AB}, \tag{3.1}$$

where ρ^{AB} is a bispinor, introduced in [1], satisfying the condition

$$\rho_{AB}\,\rho^{AB} = \bar{\rho}_{\dot{X}\dot{Y}}\,\bar{\rho}^{\dot{X}\dot{Y}} \tag{3.2}$$

which is the spinor analogue of Plücker's quadratic condition. The condition (3.2) is identically satisfied by the eigenbispinors, and if we replace the Plücker bispinor in (3.1) by u^{AB}, v^{AB} and w^{AB} respectively, then we define three lines in

$P_2(\mathbf{C})$. We then say that such a line is *proper* if the corresponding eigenbispinor of the Weyl spinor admits a non-zero eigenvalue, and *improper* if the corresponding eigenvalue is zero. The line defined by u^{AB} is denoted by u, and similar language is used for the bispinors v^{AB} and w^{AB}. Hence the Bel Petrov classification defines the following sets of lines in $P_2(\mathbf{C})$.

Type I: One proper line u

Type IIa: Two proper lines v and w

Type IIb: Two proper lines u and v

Type D: Three proper lines u, v and w

Type III: Two improper lines u and v

Type N: Two improper lines v and w.

The second approach involves interpreting a bispinor x^{AB} with components $(x^{11}, x^{12} = x^{21}, x^{22})$ as representing a point in $P_2(\mathbf{C})$ with the complex homogeneous coordinates (X^0, X^1, X^2). A conic Ω in $P_2(\mathbf{C})$ is the quadratic locus of points

$$h_{ABCD}\, x^{AB}\, x^{CD} \equiv 0 \tag{3.3}$$

where $h_{ABCD} = \varepsilon_{AC}\, \varepsilon_{BD}$. The conic Ω is non-degenerate and (3.3) is of the explicit form

$$x_{AB}\, x^{AB} = 2\,(X^0 X^2 - (X^1)^2) \neq 0\,. \tag{3.4}$$

We now assume that $P_2(\mathbf{C})$ admits a conic Ω as defined above but that x^{AB} is *not* one of the eigenbispinors of $\psi^{AB}{}_{CD}$.

The equation (2.3) may be interpreted as representing a projectivity in $P_2(\mathbf{C})$. Then an equation of the form

$$\psi^{AB}{}_{CD}\, u^{CD} = \Psi_2\, u^{AB} \tag{3.5}$$

(which occurs in Types I and D) may be regarded as defining an invariant point U of the projectivity $\psi^{AB}{}_{CD}$. Likewise an equation of the form

$$\psi^{AB}{}_{CD}\, u^{CD} = 0 \tag{3.6}$$

(which occurs in Types III) may be regarded as defining a variable point U of the projectivity $\psi^{AB}{}_{CD}$. Note that by virtue of (1.7) and (3.4) the point U lies on the conic Ω. Hence (3.5) defines an invariant point on Ω while (3.6) defines a variable point on Ω. Note that the bispinor v^{AB} also defines a point on Ω, but w^{AB} corresponds to a point not on Ω and we may regard W be either external or internal to Ω, and we chose it to be external to Ω.

These considerations allow us to give the following geometric realizations of the theorem proven in Section 2.

Type I: One invariant point U on Ω

Type IIa: Two invariant points V and W. V lies on Ω and W is external to Ω.

Type IIb: Two invariant points U and V both on Ω. Hence the eigenspinors define an invariant chord of Ω.

Type D: Three invariant points U, V and W. U and V are on Ω, W is external to Ω. Hence the eigenbispinors define an invariant triangle, one side of which is an invariant chord of Ω.

Type III: Two variable points U and V on Ω. Hence the eigenbispinors define a variable chord of Ω.

Type N: Two variable points V and W. V is on Ω, W is external to Ω.

Thus the strong form of the Bel-Petrov classification defines six distinct geometric configurations in $P_2(\mathbf{C})$. Notice by virtue of the linear independence of the bispinors in the triad $\underset{\sim}{Z}^{AB}$ none of these points can coincide.

BIBLIOGRAPHY

[1] J. D. ZUND (1969) – *Algebraic invariants and the projective geometry of spinors*, « Ann. di mat. pura ed appl », Ser. IV, *82*, 381–412.

[2] J. D. ZUND (1976) – *A memoir on the projective geometry of spinors*, « Ann. di mat. pura ed appl. », Ser. IV, *110*, 29–136.