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Antonio Lanteri, Marino Palleschi

## On k-dimensional elliptic scrolls

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**Geometria algebrica.** — *On k-dimensional elliptic scrolls* (*). Nota di ANTONIO LANTERI e MARINO PALLESCHI (**), presentata (***) dal Corrisp. E. MARCHIONNA.

RIASSUNTO. — Le varietà algebriche proiettive complesse, non singolari, di dimensione $k \geq 3$ (grado $d > 3$), a curva sezione ellittica, o sono razionali o sono fasci ellittici di spazi lineari.

Le varietà del primo tipo sono state studiate e classificate da Enriques (cfr. [2], [3]) e Scorza (cfr. [8]); alle varietà del secondo tipo è dedicata la presente Nota.

Si illustrano alcune proprietà delle varietà fibrate in spazi lineari su di una curva ellittica, e si studiano i loro modelli linearmente normali W. Indicati con $d$ e $k$ il grado e la dimensione di una siffatta W e con $n$ la dimensione del minimo spazio di appartenenza, si dimostra che $d \geq 2k+1$, $d = n-1$. Infine, assegnata la curva ellittica base, si costruisce un modello esplicito di una W del tipo considerato per una qualunque dimensione $k$ e per un qualunque grado $d \geq 2k+1$.

1. – This paper is concerned with $k$-dimensional projective irreducible and complex algebraic varieties with elliptic curve sections with special regard to $k$-dimensional elliptic scrolls.

The $k$-dimensional varieties $W \subset \mathbf{P}^n$ with elliptic curve sections were studied by Castelnuovo (cf. [1]) for $k = 2$, Enriques (cf. [2], [3]) for $k = 3$ and Scorza (cf. [8], [9]) for $k \geq 3$. Substantially they show that such varieties (but cones and cubic hypersurfaces) are either elliptic pencils of $\mathbf{P}^{k-1}$'s or rational varieties.

The rational case is deeply analyzed in the previously quoted works by Enriques and Scorza (see also [7], pp. 59–60).

The present paper is devoted to the study of the elliptic case (not developed in the classical works).

Specifically here we consider an irreducible smooth $k$-dimensional ($k \geq 3$) complex algebraic variety $W \subset \mathbf{P}^n$ with elliptic curve sections. In sec. 3 we notice that if W is neither a rational variety nor a cubic hypersurface, then it is an elliptic $k$-scroll. In sec. 4 we show that the linearly normal models of such W's are varieties of degree $d$ in $\mathbf{P}^{d-1}$ (non-hyperplane). We complete this result by proving (Theorem 4.2) that a $k$-dimensional ($k \geq 2$) linearly normal variety $W'_d \subset \mathbf{P}^n$ of degree $d$ with elliptic curve sections is an elliptic

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(**) Istituto matematico «F. Enriques» – Via C. Saldini, 50 – Milano.

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$k$-scroll if and only if $n = d - 1$. In sec. 5 we prove that an elliptic $k$-scroll ($k \geq 2$) $W_d \subset \mathbf{P}^{d-1}$ has degree $d \geq 2k + 1$, and we show that for every admissible $d$ there exists an elliptic $k$-scroll of degree $d$, by constructing an explicit model of it.

2. – From now on, the word variety will always mean irreducible complex projective algebraic variety. Such a variety of dimension $k = 1$ or $k = 2$ will be called curve or surface respectively.

Let V be a smooth variety of any dimension $k$, and $|D|$ the complete linear system associated to a divisor D on V. We shall denote by $\mathcal{O}_V$ the structural sheaf of V; by $\mathcal{O}_V(D)$ the invertible sheaf associated to D; by $h^q(\mathcal{O}_V(D)) = h^q(D)$ the complex dimension of the $q$-th cohomology vector space $H^q(V, \mathcal{O}_V(D)) = H^q(\mathcal{O}_V(D))$. In the case $k \geq 2$ consider an irreducible and smooth hypersurface S on V; it is known that there exists a divisor D′ on V, linearly equivalent to D (briefly $D' \equiv D$) which does not contain S as a component and is transversal to S. By $D \cdot S$ we denote the divisor on S (defined mod linear equivalence) which D′ cuts out on S. For $k = 2$ the symbol $(D \cdot C)$ represents the intersection index of the divisors D and C on the surface V; if $D \equiv C$, we also write $(D \cdot C) = (C^2)$.

Now let B be a smooth curve and $\mathbf{P}^r$ the $r$-dimensional complex projective space. From now on, by saying that a variety V is contained in $\mathbf{P}^r$ (or by writing $V \subset \mathbf{P}^r$) we mean that V is in $\mathbf{P}^r$ but not in any hyperplane of its.

A *k-dimensional scroll over* B (briefly a $k$-scroll) is a smooth variety V embedded in some $\mathbf{P}^r$ endowed with a morphism $\pi : V \to B$ such that $F_b = \pi^{-1}(b)$ is a $\mathbf{P}^{k-1}$, for each $b \in B$ (1). If B is an elliptic curve, such a V is said an elliptic $k$-scroll.

Finally we introduce some other notations occurring in the sequel. On a $k$-dimensional smooth variety V consider a very ample divisor H. Obviously it is always possible to choose $i$ ($i = 1, \cdots, k-1$) smooth irreducible hypersurfaces $H^1, H^2, \cdots, H^i$, belonging to $|H|$, pairwise transversal and such that the *$(k-i)$-dimensional algebraic characteristic cycle* $H_{k-i} = H^1 \cap H^2 \cap \cdots \cap H^i$ (defined by intersecting these $i$ hypersurfaces) is irreducible and smooth. We denote also by E the characteristic cycle $H_1$.

By means of well-known facts holding for complex smooth varieties, some standard cohomological calculations (see [4], sec. 2) give

*Remark 2.1.* If $k \geq 3$ and $h^1(\mathcal{O}_{H_2}) = 0$, then $h^0(\mathcal{O}_V(H)) = h^0(\mathcal{O}_{H_2}(E)) + k - 2$.

*Remark 2.2.* Suppose $k \geq 3$ and $h^1(\mathcal{O}_{H_2}) = 1$, $h^2(\mathcal{O}_{H_2}) = h^1(\mathcal{O}_{H_2}(E)) = 0$. Then $h^0(\mathcal{O}_V(H)) = h^0(\mathcal{O}_{H_2}(E)) + s$, where $0 \leq s \leq k - 2$.

(1) Remark that a $k$-scroll is a model of the projectivized Proj $(\mathcal{E})$ of a vector bundle $\mathcal{E}$ of rank $k$ over B.

3. – In [8] G. Scorza studied the $k$-dimensional varieties $W \subset \mathbf{P}^n$ with elliptic curve sections. If such a variety W is not a cone and if $\deg W > 3$, then he shows that W must be either a rational variety or an elliptic pencil of $\mathbf{P}^{k-1}$'s. Now, recalling the notations introduced in sec. 2 and denoting by $\Phi_H$ the closed immersion defined by $|H|$, this result can be restated as follows.

THEOREM 3.1. (Enriques–Scorza). – *Let* V *be a $k$-dimensional smooth variety* ($k \geq 3$) *endowed with a very ample linear system* $|H|$ *the general characteristic* 1*-cycle* E *of which is an elliptic curve. Then only the following cases can occur*:

*i*) V *is isomorphic, via* $\Phi_H$, *to a smooth cubic hypersurface of* $\mathbf{P}^{k+1}$;

*ii*) V *is a rational variety*;

*iii*) V *is isomorphic, via* $\Phi_H$, *to an elliptic $k$-scroll over* E.

Another proof, different from the original one, can be found in [4].

*Remark 3.1.* If E is an elliptic curve, the 2-dimensional characteristic cycle $H_2$ is either a rational surface or it is isomorphic to an elliptic 2-scroll (Castelnuovo [1]; for a different proof working over any algebraically closed field see also [6], Theorem 3.1). From the proof of Theorem 3.1 in [4], one sees that the former case corresponds to *i*) and *ii*), whilst the latter case corresponds to *iii*). In the former case Remark 2.1 shows that the linearly normal models $W_d = \Phi_H(V)$ of the V's occurring in case *ii*) are varieties of degree $d = (E^2)$ in $\mathbf{P}^{d+k-2}$.

For the sequel we need also the following

*Remark 3.2.* Let $W \subset \mathbf{P}^n$ be a k-scroll ($k \geq 3$) over a (smooth) curve B and let H′ be a hyperplane section of its. Then there happens: either *i*) H′ is a $(k-1)$-scroll over B, or *ii*) $H' = F_{b_1} + \cdots + F_{b_r} + S$ where $F_{b_i}$ is a fibre of W and S is a $(k-1)$-scroll over B (2), (cf. [4]).

4. – Theorem 3.1 is particularly meaningful when H is a general hyperplane section of a smooth $k$-dimensional variety $W \subset \mathbf{P}^n$. In this case it says that a smooth variety $W_d \subset \mathbf{P}^n$ of degree $d > 3$ with elliptic curve sections is either a rational variety or an elliptic $k$-scroll. The rational case (case *ii*) of Theorem 3.1) has been studied, for $k = 3$, by Enriques in [2] (see also [3]). He proved that the linearly normal models of the rational threefolds with elliptic curve sections are varieties $W_d \subset \mathbf{P}^{d+1}$ of degree $d$ ($4 \leq d \leq 8$) representable on $\mathbf{P}^3$ by a linear system of quadric or cubic surfaces. For $k > 3$, the linearly normal models in the rational case have been studied by Scorza (cf. [8]). They are, for $k = 4$, a $W_5 \subset \mathbf{P}^7$, and a $W_6 \subset \mathbf{P}^8$; for $k = 5$, a

(2) In a sense this Remark has a converse (see [4], Prop. 3.1); it could also be easily proven (cf. the following Theorem 5.1) that $1 \leq r \leq \deg W - 2k + 1$.

$W_5 \subset \mathbf{P}^8$; for $k = 6$ the Grassmannian $W_5 \subset \mathbf{P}^9$, and, for each $k \geq 3$, there is always the complete intersection $W_4 \subset \mathbf{P}^{k+2}$ of two quadrics.

This sec. is meanly devoted to the analysis of elliptic $k$-scrolls, i.e. we are dealing with the case *iii*) of Theorem 3.1.

Now consider a $k$-dimensional smooth variety $V$ ($k \geq 2$), a complete very ample linear system $|H|$ on $V$, and its characteristic cycles $H_2$ and $H_1 = E$. From now on suppose $E$ is an elliptic curve; so we know (cf. [6]) $H_2$ is either a rational surface or an elliptic geometrically ruled surface (to wit it is isomorphic to an elliptic 2-scroll). From now on suppose also we are in the latter case; so, if $k \geq 3$ all we said in Remark 3.1 implies that *the variety* $W_d = \Phi_H(V)$ *is a linearly normal elliptic* $k$*-scroll* of degree $d = (E^2)$; note that this conclusion holds also for $k = 2$ (cf. [6]). It is known (cf. [6], p. 91) that

$$h^0(\mathcal{O}_{H_2}(E)) = d \tag{4.1}$$

and $h^1(\mathcal{O}_{H_2}(E)) = 0$; moreover, as $H_2$ is supposed to be an elliptic ruled surface if $k \geq 3$, we can apply Remark 2.2 and conclude that

$$h^0(\mathcal{O}_V(H)) = d + s, \tag{4.2}$$

where $0 \leq s \leq k - 2$. So, *a priori*, $W_d$ is a linearly normal $k$-scroll of degree $d$ in $\mathbf{P}^{d+s-1}$. Really, for each $k \geq 2$ (if $k = 2$ see (4.1)), we can prove the following

THEOREM 4.1. $W_d$ *is contained in* $\mathbf{P}^{d-1}$ (*i.e.* $s = 0$).

*Proof.* In case $k = 2$ our thesis follows immediately from all we said; so we prove the theorem by induction on $k$.

*Step 1. There results* $s \leq 1$. By Remark 3.2 the general hyperplane section of $W_d \subset \mathbf{P}^{d+s-1}$ is an elliptic $(k-1)$-scroll $H_d \subset \mathbf{P}^{d+s-2}$ of degree $d$. Were $H_d$ linearly normal, the dimension of the corresponding embedding space should be $n = d - 1$, by induction; if not, it would be $n < d - 1$. So we must have $n = d + s - 2 \leq d - 1$.

*Step 2. There results* $s = 0$. By absurd suppose $s = 1$, i.e. $W_d \subset \mathbf{P}^d$. We are proving this fact can not occur by analyzing the following three cases:

$$(a)\quad d < 2k - 1\,; \qquad (b)\quad d \geq 2k\,; \qquad (c)\quad d = 2k - 1.$$

In case (*a*) the thesis is trivial; in fact two distinct fibres $F_1 = \mathbf{P}^{k-1}$ and $F_2 = \mathbf{P}^{k-1}$ of $W_d$ generate a $\mathbf{P}^{2k-1}$.

In case (*b*) the linear span $\langle F_a, F_b\rangle$ of two fibres $F_a$ and $F_b$ of $W_d$ has codimension $\geq 1$; so there exists a hyperlane $\Pi \supseteq \langle F_a, F_b\rangle$. In view of Remark 3.2, we have

$$\Pi \cap W_d = F_a + F_b + F_{a_1} + \cdots + F_{a_i} + S,$$

where $F_{a_1}, \cdots, F_{a_i}$ are fibres of $W_d$ and S is an elliptic $(k-1)$-scroll of degree $d-2-i$. Suppose S is contained in $\mathbf{P}^n$ (and not in any hyperplane of its). By induction

$$n \leq d - 2 - i - 1 . \tag{4.3}$$

Denote by $L = \langle S, F_{a_1}, \cdots, F_{a_i}, F_a \rangle$ the linear span of S and $F_{a_1}, \cdots, F_{a_i}, F_a$. Remark that S cuts out on each fibre F of $W_d$ the $\mathbf{P}^{k-2}$ which is the corresponding fibre of S; there follows that $\dim L \leq n + i + 1$; recalling (4.3), $\dim L \leq d - 2$. So there is a pencil $\{\Pi_t\}_{t \in \mathbf{P}^1}$ of hyperplanes $\Pi_t \subset \mathbf{P}^d$ through L, at least. Such a $\Pi_t$ intersects $W_d$ along

$$S + F_{a_1} + \cdots + F_{a_i} + F_a + R ,$$

where R is nothing but a fibre of $W_d$, by Remark 3.2. So it is defined a non-constant morphism from $\mathbf{P}^1$ to the elliptic base B of $W_d$. This is absurd.

In case (*c*), in order to exclude the varieties $W_d \subset \mathbf{P}^d$ $(d = 2k - 1)$ consider a hyperlane $\Pi$ containing a fibre F of $W_d$ and the corresponding section $\Pi \cap W_d = F + S$; by Remark 3.2, S is either a $(k-1)$-scroll or a $(k-1)$-scroll plus some fibres $F_{a_1}, \cdots, F_{a_i}$. The latter case does not occur: otherwise the fibres $F_{a_1}$ and F, contained in $\Pi = \mathbf{P}^{2k-2}$, should meet. Suppose S is contained in $\mathbf{P}^n$ (and not in any hyperplane of its). As S has degree $d - 1$, by induction we have $n \leq d - 2$ (3); hence there exists a pencil $\{\Pi_t\}_{t \in \mathbf{P}^1}$ of hyperplanes $\Pi_t \subset \mathbf{P}^d$ through the $\mathbf{P}^n$ containing S. Such a $\Pi_t$ intersects $W_d$ along $S + R$, where, by Remark 3.2, R is nothing but a fibre of $W_d$. Thus we can conclude, as in case (*b*).

There follows immediately

*Remark 4.1.* Let W be an elliptic $k$-scroll and H′ a general hyperplane section of its. Then H′ and each other general characteristic cycle of $|H'|$ is not linearly normal.

Let $W_d \subset \mathbf{P}^n$ be a smooth $k$-dimensional linearly normal variety $(k \geq 2)$ of degree $d$ with elliptic curve sections. If $W_d$ is a $k$-scroll, Theorem 4.1 shows that $n = d - 1$. On the other side if $n = d - 1$ and $k \geq 3$, $W_d$ can be neither a cubic hypersurface nor a rational variety (see Remark 3.1). Hence, by Theorem 3.1, $W_d$ is an elliptic $k$-scroll. The same conclusion holds if $k = 2$ (cf. [6] Corollary 3.1). So we can state

THEOREM 4.2. *Let $W_d \subset \mathbf{P}^n$ be a k-dimensional linearly normal smooth variety $(k \geq 2)$ of degree d with elliptic curve sections. Then $W_d$ is an elliptic k-scroll if and only if $n = d - 1$.*

5. – As we already said the number of possible varieties occurring in case *ii*) of Theorem 3.1 decreases as the dimension $k$ of the variety increases. On the contrary here we show that the degree $d$ of the varieties possibly

(3) Really it is $n = d - 2$; otherwise two fibres of S should meet.

occurring in case *iii*) of Theorem 3.1 must satisfy the bare condition $d \geq 2k + 1$ and that for every admissible $d$ there exists an elliptic $k$-scroll of degree $d$. This latter fact will be proved by exhibiting an explicit projective model. We start with the following

PROPOSITION 5.1. *Let $W_d \subset \mathbf{P}^{d-1}$ be an elliptic k-scroll ($k \geq 2$) of degree $d$; then $d \geq 2k + 1$.*

*Proof*. This fact is trivial for $k = 2$. Let us continue the proof by induction on $k$. By absurd, let $W_d \subset \mathbf{P}^{d-1}$ be an elliptic $k$-scroll of degree

$$d < 2k + 1 . \tag{5.1}$$

Consider a fibre F of $W_d$, a hyperplane $\Pi$ containing F and the corresponding section

$$\Pi \cap W_d = F + S . \tag{5.2}$$

By reasoning as in the proof of Theorem 4.1 (Step 2, case (*c*)) we conclude that S is an elliptic $(k-1)$-scroll of degree $d-1$. Consider the linearly normal $(k-1)$-scroll S′ of degree $d-1$ embedded by the complete linear system of the hyperplane sections of S. By Theorem 4.2, S′ is contained in $\mathbf{P}^{d-2}$; so, by induction, $d - 1 \geq 2(k-1) + 1$. Recalling (5.1) it can only be $d = 2k$. In this case consider the $(k-1)$-scroll S in (5.2) and let L be a $\mathbf{P}^{d-k} = \mathbf{P}^k$ containing the fibre F. By applying repeatedly Remark 3.2 we see that $L \cap S$ contains an elliptic curve C, isomorphic to B, which must be a section of $W_d$ (in the sense that C intersects each fibre in one point only). But as F is a hyperplane in L, $\mathrm{Card}(C \cap F) = \deg C$, and $\deg C > 1$, C being elliptic.

The bound given by Proposition 5.1 is as best as possible. In fact, for each $k \geq 2$, we are going to construct an elliptic $k$-scroll of degree $d = 2k + 1$. This construction generalizes the one given in [5] for the quintic elliptic 2-scroll in $\mathbf{P}^4$.

In $\mathbf{P}^{2k}$ consider $k$ distinct 2-planes $\pi_i$ $(i = 1, \cdots, k)$ pairwise intersecting in a single common point $p_0$ and spanning the whole $\mathbf{P}^{2k}$. Consider also an elliptic curve B and $k$ distinct points $b_1, \cdots, b_k$ of B such that each $b_i - b_j$ $(i, j = 1, \cdots, k; i \neq j)$ is not of order three, i.e.

$$3(b_i - b_j) \neq 0 \qquad (i, j = 1, \cdots, k; i \neq j) . \tag{5.3}$$

Let $\eta_i : B \to \pi_i$ $(i = 1, \cdots, k)$ be a closed immersion such that $\eta_i(b_i) = p_0$, $(i = 1, \cdots, k)$, and denote by $B_i$ the elliptic cubic curve $\eta_i(B)$. For each $b \in B$, $b \neq b_i$ consider the $\mathbf{P}^{k-1}$

$$F_b = \langle \eta_1(b), \eta_2(b + b_2 - b_1), \cdots, \eta_k(b + b_k - b_1) \rangle \tag{5.4}$$

spanned by the independent points $\eta_i(b + b_i - b_1) \in B_i$ $(i = 1, \cdots, k)$. Consider also the map

$$\Phi : B \setminus \{b_1\} \to \text{Grass}(k-1, 2k)$$

which takes values in the Grassmann manifold of the $\mathbf{P}^{k-1}$'s in $\mathbf{P}^{2k}$, defined by $\Phi(b) = F_b$. Denote by $\overline{\Phi}$ its extension to B and put

$$F_{b_1} = \overline{\Phi}(b_1). \tag{5.5}$$

LEMMA 5.1. *If* $b, b' \in B$ *are two distinct points, then* $F_b \cap F_{b'} = \varnothing$.

*Proof.* *i*) *Suppose* $b, b' \in B \setminus \{b_1\}$. Put $p_i = \eta_i(b + b_i - b_1)$ and $p_i' = \eta_i(b' + b_i - b_1)$, $(i = 1, \cdots, k)$. There results

$$F_b = \langle p_1, \cdots, p_k \rangle \quad \text{and} \quad F_{b'} = \langle p_1', \cdots, p_k' \rangle .$$

By absurd, suppose $F_b \cap F_{b'} \neq \varnothing$; there follows

$$\dim \langle F_b, F_{b'} \rangle \leq 2k - 2 . \tag{5.6}$$

Let $l_i$ be the line $\langle p_i, p_i' \rangle$ and suppose such lines are pairwise skew. The linear span $S_{1,2} = \langle l_1, l_2 \rangle$ has dimension 3. Moreover $S_{1,2} \subset \langle \pi_1, \pi_2 \rangle$; so $S_{1,2} \cap l_3$, which is contained in $\langle \pi_1, \pi_2 \rangle \cap \pi_3 = \{p_0\}$, is either empty or reduced to $p_0$. But the latter case can not happen and so

$$S_{1,2} \cap l_3 = \varnothing . \tag{5.7}$$

In fact, if $S_{1,2} \cap l_3 = \{p_0\}$, then there were two lines $l_3$ and $l_1$ (or $l_2$) through $p_0$, contradicting our assumption; to see this it is sufficient to prove that if $S_{1,2} \ni p_0$ then $l_1$ (or $l_2$) itself contains $p_0$. In fact, as $\dim S_{1,2} = 3$, $S_{1,2}$ cannot contain $\pi_1 \cup \pi_2$. So it cuts out on one of them (4), suppose $\pi_1$, the line $l_1$. But $l_1$ intersects the cubic curve $B_1$ in $p_1, p_1'$ and in a further point which must be $p_0$, as $p_0 \in S_{1,2}$; hence $p_0 \in l_1$. Thus (5.7) is true and then $S_{1,2,3} = \langle S_{1,2}, l_3 \rangle = \langle l_1, l_2, l_3 \rangle$ has dimension 5. By repeating the same argument we conclude

$$\dim \langle l_1, l_2, \cdots, l_k \rangle = 2k - 1 . \tag{5.8}$$

As $\langle F_b, F_{b'} \rangle = \langle l_1, \cdots, l_k \rangle$, (5.8) contradicts (5.6). Thus the lines $l_i$ cannot be pairwise skew. Suppose $l_i \cap l_j \neq \varnothing$; it must be $l_i \cap l_j = \{p_0\}$, and so the triple $p_0, p_i, p_i'$ (respectively $p_0, p_j, p_j'$) is collinear on $B_i$ (respectively on $B_j$). This means that the pair $b + b_i - b_1$, $b' + b_i - b_1$ must belong to the

(4) It could be seen (cf. the following part of this proof) that if $p_0 \in S_{1,2}$ and $S_{1,2} \not\supset \pi_1$, then $S_{1,2} \supset \pi_2$.

$g_2^1$ defined by $b_i$ on B and that the pair $b + b_j - b_1$, $b' + b_j - b_1$ must belong to the $g_2^1$ defined by $b_j$ on B. Therefore it must be

$$b + b_i - b_1 + b' + b_i - b_1 + b_i = 0 \, , b + b_j - b_1 + b' + b_j - b_1 + b_j = 0 \, .$$

But such relations cannot hold together by the assumption (5.3).

*ii) Suppose* $b' = b_1$. Assuming, by absurd, $F_b \cap F_{b'} \neq \varnothing$, reasoning similarly to the case *i)* and taking into account (5.5), one sees that the relations

$$b + b_i - b_1 + 2\, b_i = 0 \quad , \quad b + b_j - b_1 + 2\, b_j = 0$$

must hold together for some $i$ and $j$ $(i \neq j)$. Again this contradicts (5.3).

Now put

$$W = \bigcup_{b \in B} F_b \, ; \tag{5.9}$$

there holds the following

PROPOSITION 5.2. W *is an elliptic k-scroll over* B *embedded in* $\mathbf{P}^{2k}$ *and of degree* $d = 2\,k + 1$.

*Proof.* By construction and Lemma 5.1 it follows immediately that (5.9) defines an elliptic $k$-scroll (over B) $W \subset \mathbf{P}^{2k}$; we prove by induction that it has degree $d = 2\,k + 1$. For $k = 2$, W is the quintic elliptic 2-scroll in $\mathbf{P}^4$ (cf. [5], Prop. 5.1). Otherwise consider a hyperplane $\Pi \subset \mathbf{P}^{2k}$ containing the planes $\pi_1, \cdots, \pi_{k-1}$. The hyperplane $\Pi$ cuts out on the cubic curve $B_k$ the point $p_0$ and two other points $p_1$ and $p_2$. Hence $W \cap \Pi$ is constituted by the two fibres of W through $p_1$ and $p_2$ and by the $(k - 1)$-scroll S generated by the cubic curves $B_1, \cdots, B_{k-1}$ (in the same way as W is generated). So $\deg W = 2 + \deg S$, and by induction on S we conclude.

The previous construction can be generalized and gives a model of an elliptic $k$-scroll of degree $d$ for each $d \geq 2\,k + 1$. Replace the 2-planes $\pi_i$ $(i = 1, \cdots, k)$ by linear spaces $L_i$ of dimensions $r_i\,(r_i \geq 2)$ contained in $\mathbf{P}^r$ $(r = \Sigma\, r_i)$, pairwise intersecting in a single common point $p_0$ and generating the whole $\mathbf{P}^r$. Replace the cubic curves $B_i$ by elliptic curves of degree $r_i + 1$ isomorphic to B via closed immersions $\eta_i$. Define the $(k - 1)$-linear space $F_b$ as in (5.4). Lemma 5.1 continues to hold and so *a formula analogous to* (5.9) *defines a variety* W *which is easily seen to be an elliptic k-scroll over* B *of degree* $d = \sum_{i=1}^{k} r_i + 1 = r + 1$ *contained in* $\mathbf{P}^r$.

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