Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On a theorem of Tate

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **69** (1980), n.3-4, p. 117–119.

Accademia Nazionale dei Lincei

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Algebra. — *On a theorem of Tate.* Nota (*) di GABRIEL CHIRIACESCU, presentata dal Socio E. MARTINELLI.

RIASSUNTO. — In quest'articolo si dimostra una generalizzazione di un teorema di Tate ([11] 6.3, [5] Teorema 69); come corollario si ottiene un teorema di Marot ([3]) dal quale si ricava una proprietà di stabilità degli anelli universalmente giapponesi nel passaggio al completamento I–adico.

0. INTRODUCTION

In this note we use the notations and terminology from: [EGA] or [9].

The universally japanese rings were introduced and studied by Nagata in [6]. In [3] Marot proved the stability of these rings to the adic completion. Since in the local case " universally japanese " is equivalent to " the formal fibers are geometrically reduced ", this theorem gives a positive answer to a conjecture of A. Grothendieck: [EGA] (7.4.8).

In [11] Tate proved the following:

THEOREM 0.1. *If* A *is a noetherian normal domain and* $0 \neq x$ *a prime element, such that* A *is* xA*-adically complete and* A/xA *is japanese, then* A *is japanese.*

From this theorem they deduce easy that any local, complete, noetherian domain is japanese. (cf. [5] 31.C).

In [10] Seydi showed that the normality condition in (0.1) is not necessary. In [4] Marot dropped out the condition " x a prime element ", asking that A/P to be japanese for any $P \in \mathrm{Ass}(A/xA)$ but imposed restrictive conditions on the integral closure of A. The proofs from: [EGA], [5], [10], and [4] use essentially the characteristic of A.

In [8] (Corollary 4) Nishimura gave an entirely new proof of the statement from [10], as a corollary of his noetherianity criterion for Krull domains (see [8]).

In this note we present a generalisation of Tate's theorem, stronger than the above ones, since it has as corollary the Marot's theorem ([3]). The method of the proof uses essentially the Nishimura's noetherianity criterion.

(*) Pervenuta all'Accademia il 18 ottobre 1980.

1. The following proposition is another form of the Nishimura's criterion, and is contained in its proof:

PROPOSITION 1.1 *Let* A *be a Krull domain and let* $0 \neq x \in A$ *such that,* A/P *is noetherian, for every minimal prime* P *over* xA. *Then* A/xA *is noetherian.*

The main result of this note is the following:

THEOREM 1.2 *Let* A *be a noetherian domain and* x *an element in* A. *Suppose that*:

i) *For all prime* $P \in \mathrm{Ass}(A/xA)$, A/P *is japanese.*

ii) A *is* xA*-adically complete and separated.*

Then A *is japanese.*

Proof. Let K′ be a finite extension of the field of quotients K of A. Let $\bar{A}$ be the integral closure of A in K and A′ be the integral closure of A in K′. We want to show that A′ is finite over A.

Let $\{Q_1, \cdots, Q_n\}$ be the set of all the minimal primes over xA'. Let $\bar{P}_i = Q_i \cap \bar{A}$ and $P_i = Q_i \cap A$. By going-down theorem we deduce that $ht(\bar{P}_i) = 1$ and by [7] (33.11) it follows that $P_i \in \mathrm{Ass}(A/xA)$. Since $[k(Q_i) : k(P_i)] < \infty$ (cf. [7] (33.10)) (where $k(Q_i)$ is the field of quotients of A/Q_i) and using *i*) we deduce that A'/Q_i is finite over A/P_i. In particular A'/Q_i is noetherian, $1 \leq i \leq n$, and by (1.1) it follows that A'/xA' is noetherian (since A′ is a Krull domain, cf. [7] (33.10)). Let J be the radical of xA'. It's easy to see that A'/J is finite over A/xA; since A'/xA' is noetherian there exists $n \in N$ such that $J^n \subseteq xA'$. Applying the induction in the following exact sequence:

$$0 \to J^n/J^{n+1} \to A'/J^{n+1} \to A'/J^n \to 0$$

we deduce that A'/xA' is finite over A/xA. Since A is xA-adically complete and A′ is xA'-adically separated, a well-known lemma leads us to the fact that A′ is finite over A.

COROLLARY (1.3) (Marot). *Let* A *be a noetherian ring and* I *an ideal in* A. *Suppose that*:

i) A *is* I*-adically complete and separated.*

ii) A/I *is universally japanese.*

Then A *is universally japanese.*

Proof. Using the induction on the number of generators of I we may suppose that I is principal: $I = xA$. Take $Q \in \mathrm{Spec}(A)$: if $xA \subseteq Q$ then A/Q is japanese by *ii*); if $xA \nsubseteq Q$ let $B = A/Q$. Take $P \in \mathrm{Ass}(B/xB)$. Then there exists $P' \in \mathrm{Spec}(A)$, such that $xA \subseteq P'$ and $A/P' \cong B/P$. By *ii*) we deduce that B/P is japanese, hence B is japanese by (1.2), hence A is universally japanese.

The statement of the Marot's theorem and Theorem (1.2) lead to the following:

OPEN QUESTION. *Suppose* A *is noetherian domain and* I *is an ideal in* A *such that*:

i) A *is* I-*adically complete and separated.*

ii) A/P *is japanese for every* P ∈ Ass (A/I).

Does it follow that A *is japanese?*

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