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The plate on unilateral elastic boundary support. Nota I

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Meccanica dei solidi. — *The plate on unilateral elastic boundary support* (*). Nota I di RAFFAELE TOSCANO (**) e ALDO MACERI (***), presentata (****) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — Si studia il problema della piastra elastica con appoggio elastico unilaterale al bordo. Si danno risultati di esistenza e unicità della soluzione.

We consider the problem of the linearly elastic plate, under transverse loads, resting on elastic, unilateral boundary support.

Given the bounded and connected domain Ω occupied by the plate in its middle plane $x_1 x_2$, let us assume external forces q and displacements v to be positive in x_3 direction (the orthogonal reference frame $Ox_1 x_2 x_3$ is anticlockwise).

The reaction r of the edge constraint has a " Winkler type " expression:

$$r = -\mathrm{E}v^+$$

where E is a non-negative function.

It is convenient to formulate the elastic equilibrium problem like an energetic one, considering a sufficiently general fourth order operator and taking into account distributed and/or concentrated forces.

Hence, we let:

Ω a bounded and connected open of R^2 of class $\mathbf{R}^{(0),1}$ (in symbols $\Omega \in \mathbf{R}^{(0),1}$ [1]),

Γ the boundary of Ω,

s the curvilinear measure on Γ [1],

$$\mathrm{A} = \sum_{\substack{|r|=2\\|s|=2}} \mathrm{D}^r (a_{rs}\, \mathrm{D}^s), \quad \text{with} \quad a_{rs} \in \mathrm{L}^\infty(\Omega) \quad \text{and} \quad a_{rs} = a_{sr},$$

a fourth order differential operator such that:

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs}\, \mathrm{D}^s v\, \mathrm{D}^r v \, \mathrm{d}x \geq a_0 \sum_{|r|=2} \int_\Omega |\mathrm{D}^r v|^2 \, \mathrm{d}x \;\; \forall v \in \mathrm{W}^2(\Omega)$$

$$(a_0 = \text{const.} > 0),$$

$$\mathrm{E} \in \mathrm{L}^\infty(\Gamma) - \{0\}, \quad \text{with} \quad \mathrm{E} \geq 0 \quad s\text{-a.e. on } \Gamma, \quad q \in (\mathrm{W}^2(\Omega))'.$$

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Furthermore, we let $\forall v \in W^2(\Omega)$:

$$J(v) = \frac{1}{2} \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s v D^r v \, dx + \frac{1}{2} \int_\Gamma E [v^+]^2 \, ds - \langle q , v \rangle ,$$

and we are concerned with the following total potential energy minimum problem:

PROBLEM (P). *Find* $u \in W^2(\Omega)$ *such that*:

$$J(u) \leq J(v) \qquad \forall v \in W^2(\Omega) .$$

In N. 1 we will give some formulations equivalent to problem (P), in N. 2 we will study solution's existence and uniqueness questions, whose regularity will be finally analyzed in N. 3 of Note II.

1. – LEMMA 1. *The functional* J *is convex, Gateaux-differentiable in* $W^2(\Omega)$ *and results*:

$$J'(u , v) = \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s u D^r v \, dx + \int_\Gamma E u^+ v \, ds - \langle q , v \rangle$$

$$\forall (u , v) \in (W^2(\Omega))^2 .$$

Consequently J *is weakly lower semicontinuous on* $W^2(\Omega)$.

Proof. Convexity is obvious. Let us prove that J is differentiable. It is sufficient to prove that the functional:

$$v \in W^2(\Omega) \to \frac{1}{2} \int_\Gamma E (v^+)^2 \, ds$$

is differentiable. The Lebesgue theorem on dominated convergence applies. □

LEMMA 2. *For any* $u \in W^2(\Omega)$, *the functional* $J'(u , \cdot)$ *is linear and continuous on* $W^2(\Omega)$. *Moreover the operator*:

$$B : u \in W^2(\Omega) \to J'(u , \cdot)$$

is monotone and hemicontinuous.

Proof. Linearity of $J'(u , \cdot)$ is obvious. As for continuity, it is obviously sufficient to prove it for the functional:

$$F : v \in W^2(\Omega) \to \int_\Gamma E u^+ v \, ds .$$

Because $\Omega \in \mathbf{R}^{(0),1}$, we have [1]:

$$\text{(1)} \qquad \| v \|_{L^2(\Gamma)} \leq \text{const.} \, \| v \|_{W^1(\Omega)} \qquad \forall v \in W^2(\Omega) .$$

Continuity of F is then acquired by observing that:

(2) For any u and v elements of $W^2(\Omega)$ it results:

$$\int_\Gamma | E u^+ v | \, ds \leq \text{const.} \, \| E \|_{L^\infty(\Gamma)} \cdot \| u^+ \|_{L^2(\Gamma)} \cdot \| v \|_{L^2(\Gamma)} .$$

Monotonicity and hemicontinuity of B are obvious. □

By using Lemma 1, we easily prove that:

THEOREM 1. *For any* $u \in W^2(\Omega)$, *the following statements are equivalent*:

a) *u is a solution of problem* (P),

b) *u is a solution of the variational (virtual work) equation*:

$$\text{(3)} \qquad u \in W^2(\Omega) : \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s u \, D^r v \, dx + \int_\Gamma E u^+ v \, ds = \langle q , v \rangle \qquad \forall v \in W^2(\Omega) .$$

c) *u is a solution of the mixed type variational inequality*:

$$u \in W^2(\Omega) : \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s u \, D^r (v - u) \, dx - \langle q , v - u \rangle + \frac{1}{2} \int_\Gamma E (v^+)^2 \, ds - \frac{1}{2} \int_\Gamma E (u^+)^2 \, ds \geq 0 \qquad \forall v \in W^2(\Omega) .$$

2. – Let us study now existence and uniqueness of the problem (P) solution.

Let us note as P_1 the subspace of $W^2(\Omega)$ of the not greater than 1st degree polynomials, and let us recall that, because $\Omega \in \mathbf{R}^{(0),1}$, it results [1]:

$$\text{(4)} \qquad c_1' \left(\sum_{|r|=2} \int_\Omega | D^r v |^2 \, dx \right)^{\frac{1}{2}} \leq \| \bar{v} \|_{\frac{W^2(\Omega)}{P_1}} \leq c_1 \left(\sum_{|r|=2} \int_\Omega | D^r v |^2 \, dx \right)^{\frac{1}{2}} \qquad \forall \bar{v} = [v] \in \frac{W^2(\Omega)}{P_1} ,$$

where the positive constants c_1 and c_1' are independent of v.

We let $\Gamma_E = \{x \in \Gamma \mid E(x) > 0\}$ and, $\forall x = (x_1, x_2) \in R^2$:

$$\mathbf{1}(x) = 1 \quad , \quad p_1(x) = x_1 \quad , \quad p_2(x) = x_2 .$$

Moreover, if $\langle q, \mathbf{1} \rangle > 0$, we let:

$$\xi = \left(\frac{\langle q, p_1 \rangle}{\langle q, \mathbf{1} \rangle}, \frac{\langle q, p_2 \rangle}{\langle q, \mathbf{1} \rangle} \right),$$

and we remark that $\langle q, \mathbf{1} \rangle$ is the component in the x_3 direction of the external forces resultant, applied, if nonzero, at ξ.

THEOREM 2. *If problem* (P) *has solution, then* $\langle q, \mathbf{1} \rangle \geq 0$. *If* $\langle q, \mathbf{1} \rangle = 0$ *and problem* (P) *has solution, then* $\langle q, p_1 \rangle = \langle q, p_2 \rangle = 0$. *If* $\langle q, \mathbf{1} \rangle > 0$ *and problem* (P) *has solution, then*:

$$\text{(5)} \qquad \forall p \in P_1 - \{0\}, \quad \text{with} \quad p(\xi) = 0, \\ s(\{x \in \Gamma_E \mid p(x) \geq 0\}) > 0 .$$

If $\langle q, \mathbf{1} \rangle > 0$ *and*:

$$\text{(6)} \qquad \exists p_0 \in P_1 - \{0\}, \quad \textit{with} \quad p_0(\xi) = 0 \quad , \quad \ni' s(\{x \in \Gamma_E \mid p_0(x) > 0\}) = 0 ,$$

and if problem (P) *has solution, then* (5) *is true and*:

$$\text{(7)} \qquad \forall p \in P_1, \quad \textit{with} \quad p(\xi) = 0 \quad \textit{and} \quad p \neq \lambda p_0 \quad \forall \lambda \in R, \\ s(\{x \in \Gamma_E \mid p_0(x) = 0 \ , \ p(x) > 0\}) > 0 .$$

Proof. Let problem (P) admit a solution u. Because u satisfies (3), we must have:

$$\text{(8)} \qquad \int_\Gamma E u^+ \, ds = \langle q, \mathbf{1} \rangle ,$$

so that $\langle q, \mathbf{1} \rangle \geq 0$.

If $\langle q, \mathbf{1} \rangle = 0$, from (8) and from the equality:

$$\int_\Gamma E u^+ p_i \, ds = \langle q, p_i \rangle$$

follows $\langle q, p_i \rangle = 0$.

If $\langle q, \mathbf{1} \rangle > 0$, because (8) is true, we have:

$$s(\{x \in \Gamma_E \mid u(x) > 0\}) > 0 .$$

Hence, for any $p \in P_1 - \{0\}$ with $p(\xi) = 0$, because:

$$\int_\Gamma E u^+ p \, ds = \langle q, p \rangle = p(\xi) \langle q, \mathbf{1} \rangle = 0$$

it is obvious that:

$$s(\{x \in \Gamma_E \mid p(x) \geq 0\}) > 0 .$$

Let us assume now $\langle q , \mathbf{1} \rangle > 0$ and let (6) be true.

At first, because:

$$\int_\Gamma E u^+ p_0 \, ds = \langle q , p_0 \rangle = 0$$

and, by (6):

$$E u^+ p_0 \leq 0 \qquad s\text{-a.e. on } \Gamma ,$$

we have:

$$E u^+ p_0 = 0 \qquad s\text{-a.e. on } \Gamma . \tag{9}$$

After that let, by absurd, $\tilde{p} \in P_1$, with $\tilde{p}(\xi) = 0$ and $\tilde{p} \neq \lambda p_0 \ \forall \lambda \in R$, exist such that:

$$s\{x \in \Gamma_E \mid p_0(x) = 0 \ , \ \tilde{p}(x) > 0\}) = 0 . \tag{10}$$

From (9) and (10) we have:

$$E u^+ \tilde{p} \leq 0 \qquad s\text{-a.e. on } \Gamma . \tag{11}$$

Then, because u is solution of (3), we must have:

$$\int_\Gamma E u^+ \tilde{p} \, ds = \langle q , \tilde{p} \rangle = 0 . \tag{12}$$

From (11) and (12) it follows:

$$u^+ \tilde{p} = 0 \qquad s\text{-a.e. on } \Gamma_E .$$

Hence, taking account of (9) and observing that:

$$\{x \in R^2 \mid \tilde{p}(x) = p_0(x) = 0\} = \{\xi\} ,$$

results:

$$u^+ = 0 \qquad s\text{-a.e. on } \Gamma_E$$

and, consequently:

$$\int_\Gamma E u^+ \, ds = 0 .$$

But that is absurd, because:

$$\int_{\Gamma} \mathrm{E}u^{+} \, \mathrm{d}s = \langle q , \mathbf{1} \rangle > 0 .$$

Hence (7) is true. □

From Theorem 2 follows that problem (P) can allow a solution only in the following cases:

α) $\langle q , \mathbf{1} \rangle = 0 , \quad \langle q , p_1 \rangle = \langle q , p_2 \rangle = 0$;

β) $\langle q , \mathbf{1} \rangle > 0$ *and results*:

$$\forall p \in \mathrm{P}_1 - \{0\} , \quad \textit{with} \quad p(\xi) = 0 \quad , \quad s(\{x \in \Gamma_{\mathrm{E}} \mid p(x) > 0\}) > 0 ; \tag{13}$$

γ) $\langle q , \mathbf{1} \rangle > 0$ *and* (5), (6), (7) *are true*;

i.e., with different terminology, in following cases:

α) the external forces system is self-equilibrated;

β) the external forces resultant has the direction of the positive x_3 axis and is applied at a point ξ such that any through it straight line leaves on left and on right a set of constrained points whose measure is positive;

γ) the external forces resultant has the direction of the positive x_3 axis and is applied at a point ξ such that any through it straight line leaves on the right or on the same straight line (and on left or on the same straight line) a set of constrained points whose measure is positive. Moreover a straight line r exists of equation $p_0(x) = 0$ such that on its right (or its left) the set of the constrained points has measure zero and such that all straight line through ξ different from it leaves on right and on left a set of points of r whose measure is positive.

Remark 1. Let us notice that from the Proof of (7) it follows that any possible solution u of problem (P) in the γ) case is such that:

$$\mathrm{E}u^{+} p_0 = 0 \qquad s\text{-a.e. on } \Gamma . \tag{14}$$

THEOREM 3. *In the α) case, problem (P) allows infinite solutions, whose set coincides with the set of solutions of the variational equation*:

$$u \in \mathrm{W}^2(\Omega) : \sum_{\substack{|r|=2 \\ |s|=2}} \int_{\Omega} a_{rs} \, \mathrm{D}^s u \, \mathrm{D}^r v \, \mathrm{d}x = \langle q , v \rangle \qquad \forall v \in \mathrm{W}^2(\Omega) \tag{15}$$

(relative to a free plate problem) non-positive on Γ_{E}.

In the β) *case problem* (P) *allows at least a solution.*

Proof. About the α) case, by using (4), we verify immediately that (15) allows infinite solutions, whose set is an element of $\dfrac{\mathrm{W}^2(\Omega)}{\mathrm{P}_1}$.

Thus the thesis is easily proven. About the β) case, using again the problem (P) equivalence with (3), and taking account of Lemma 2, it is sufficient [2] to prove that:

$$(16) \qquad \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs}\, D^s v\, D^r v\, dx - \langle q , v\rangle + \int_\Gamma E\,(v^+)^2\, ds \to +\infty$$

$$\text{as} \quad \| v \|_{W^2(\Omega)} \to +\infty .$$

By absurd, $k > 0$ and a sequence $\{v_n\}$ of elements of $W^2(\Omega)$ exist such that:

$$(17) \qquad \| v_n \|_{W^2(\Omega)} > n \qquad \forall n \in N ,$$

$$(18) \qquad \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs}\, D^s v_n\, D^r v_n\, dx + \int_\Gamma E\,(v_n^+)^2\, ds \le \langle q , v_n\rangle + k \qquad \forall n \in N .$$

By putting $w_n = \dfrac{v_n}{\| v_n \|_{W^2(\Omega)}}$, we have, from (18):

$$a_0 \sum_{|r|=2} \int_\Omega | D^r w_n |^2\, dx \le \frac{1}{\| v_n \|_{W^2(\Omega)}} \cdot \| q \|_{(W^2(\Omega))'} + \frac{k}{\| v_n \|^2_{W^2(\Omega)}} \qquad \forall n \in N ,$$

from which:

$$(19) \qquad \text{for} \quad | r | = 2 \qquad \lim_{n \to +\infty} \| D^r w_n \|_{L^2(\Omega)} = 0 .$$

Because $\| w_n \|_{W^2(\Omega)} = 1 \quad \forall n \in N$, there exists a subsequence of $\{w_n\}$, which we denote with the same symbol, weakly-convergent in $W^2(\Omega)$ (and hence strongly in $W^1(\Omega)$) towards an element w. From this, from (18) and because the functional:

$$v \in W^2(\Omega) \to \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs}\, D^s v\, D^r v\, dx + \int_\Gamma E\,(v^+)^2\, ds$$

is weakly lower-semicontinuous we have:

$$a_0 \sum_{|r|=2} \int_\Omega | D^r w |^2\, dx + \int_\Gamma E\,(w^+)^2\, ds = 0$$

and hence:

$$\text{for} \quad | r | = 2 \qquad D^r w = 0 \quad , \qquad E w^+ = 0 \qquad s\text{-a.e. on } \Gamma .$$

Hence:

$$w \in P_1 \tag{20}$$

$$s(\{x \in \Gamma_E \mid w(x) > 0\}) = 0\,. \tag{21}$$

From (19) and (20), and because

$$\lim_{n \to +\infty} \| w_n - w \|_{W^1(\Omega)} = 0\,,$$

we have:

$$\lim_{n \to +\infty} \| w_n - w \|_{W^2(\Omega)} = 0\,,$$

and this, because $\| w_n \|_{W^2(\Omega)} = 1 \quad \forall n \in N$, implies:

$$w \neq 0\,. \tag{22}$$

Let us observe now that, from (18):

$$\langle q\,, w \rangle \geq 0\,,$$

from which, because $\langle q\,, w \rangle = w(\xi)\ \langle q\,, \mathbf{1} \rangle$:

$$w(\xi) \geq 0\,.$$

Moreover, if $w(\xi) = 0$, from (20), (22) and (13) we obtain:

$$s(\{x \in \Gamma_E \mid w(x) > 0\}) > 0\,,$$

and this contrasts with (21). Hence:

$$w(\xi) > 0\,. \tag{23}$$

Let us prove that (23) is false. To see this, we let, $\forall x \in R^2$, $Q(x) = w(x) - w(\xi)$.

If $Q = 0$, because $\forall x \in R^2 \quad w(x) = w(\xi) > 0$, we have:

$$s(\{x \in \Gamma_E \mid w(x) > 0\}) = s(\Gamma_E) > 0\,,$$

which contrasts with (21).

If $Q \neq 0$, because $Q(\xi) = 0$, from (13) we have:

$$s(\{x \in \Gamma_E \mid Q(x) > 0\}) > 0$$

which implies:

$$s(\{x \in \Gamma_E \mid w(x) > 0\}) > 0\,,$$

and this is impossible by (21). Hence (23) is false. This absurd proves (16). □

About solution existence in the γ) case, it is convenient to study an auxiliar problem. We fix on Ω a point x_1 such that $p_0(x_1) \neq 0$, and we let:

$$V_1 = \{v \in W^2(\Omega) \mid v(x_1) = 0\}.$$

Because $W^2(\Omega) \subseteq C^0(\overline{\Omega})$ with continuous imbedding, V_1, equipped by the norm of $W^2(\Omega)$, is a closed subspace of $W^2(\Omega)$. Now, we let s-a.e. on Γ:

$$E_1(x) = \begin{cases} E(x), & \text{if} \quad p_0(x) \geq 0 \\ 0, & \text{if} \quad p_0(x) < 0, \end{cases}$$

and we consider the variational equation:

$$(24) \qquad u_1 \in V_1: \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s u_1 D^r v \, dx + \int_\Gamma E_1 u_1^+ v \, ds = \langle q, v \rangle \qquad \forall v \in V_1$$

describing the elastic equilibrium of a plate supported only along r in the same way as the given plate, and moreover with imposed displacement equal to zero at x_1.

THEOREM 4. *In the hypotheses of the* γ) *case*, (24) *allows unique solution.*

Proof. About the existence of a solution of (24) it is sufficient, as already done for Theorem 3, to prove that:

$$(25) \qquad \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s v D^r v \, dx + \int_\Gamma E_1 (v^+)^2 \, ds - \langle q, v \rangle \to +\infty$$

$$\text{as} \quad \|v\|_{W^2(\Omega)} \to +\infty \qquad \text{on} \quad V_1.$$

Denying (25), in a similar way as for Theorem 3 the existence of a $w \in V_1$ is proven such that:

$$(26) \qquad w \in P_1 - \{0\}$$

$$(27) \qquad s(\{x \in \Gamma \mid E_1(x) > 0, \; w(x) > 0\}) = 0$$

$$(28) \qquad \langle q, w \rangle \geq 0.$$

Let us prove that:

$$(29) \qquad w(\xi) = 0.$$

Because by (28):

$$w(\xi) \langle q, \mathbf{1} \rangle \geq 0,$$

we must have $w(\xi) \geq 0$. By absurd, let us suppose $w(\xi) > 0$. We put, $\forall x \in R^2$, $Q(x) = w(x) - w(\xi)$ and at first we remark that:

$$Q \in P_1 - \{0\} \quad , \quad Q(\xi) = 0 .$$

Consequently, from (5), (6) and (7):

$$s(\{x \in \Gamma_E \mid p_0(x) = 0 \ , \ Q(x) \geq 0\}) > 0$$

and consequently:

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ Q(x) \geq 0\}) > 0 . \tag{30}$$

(30) implies:

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ w(x) > 0\}) > 0 ,$$

which contrasts with (27). Hence (29) is true. Now let us observe that, because $w \neq 0$, $w(x_1) = 0$ and $p_0(x_1) \neq 0$, it results:

$$w \neq \lambda p_0 \qquad \forall \lambda \in R$$

and hence, taking account of (26), (29) and (7):

$$s(\{x \in \Gamma_E \mid p_0(x) = 0 \ , \ w(x) > 0\}) > 0$$

i.e.:

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ p_0(x) = 0 \ , \ w(x) > 0\}) > 0$$

which is absurd by (27). Thus (25) is proven; consequently at least one solution u_1 of (24) exists. Now let us prove that u_1 is the unique solution of (24). By absurd, let u_2 be a solution of (24) different from u_1. Putting $\tilde{p} = u_2 - u_1$, by obvious relations:

$$\sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s (u_2 - u_1) D^r (u_2 - u_1) dx + \int_\Gamma E_1 (u_2 - u_1)(u_2^+ - u_1^+) ds = 0 ,$$

$$E_1 (u_2 - u_1)(u_2^+ - u_1^+) \geq 0 ,$$

and because $\tilde{p}(x_1) = 0$ and $p_0(x_1) \neq 0$, we have:

$$\tilde{p} \in P_1 \quad , \quad \tilde{p} \neq \lambda p_0 \qquad \forall \lambda \in R \tag{31}$$

$$u_1^+ = u_2^+ \qquad s\text{-a.e. on} \quad \{x \in \Gamma \mid E_1(x) > 0\} . \tag{32}$$

Let us now notice that, from (5) and (6):

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ p_0(x) > 0\}) = 0 , \tag{33}$$

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ p_0(x) = 0\}) > 0 ,$$

and, from (32):

$$u_1^+ = u_2^+ \qquad s\text{-a.e. on } \{x \in \Gamma \mid E_1(x) > 0 \ , \ p_0(x) = 0\} .$$

Hence, taking account of (31), we must have:

$$s(\{x \in \Gamma \mid E_1(x) > 0 \ , \ p_0(x) = 0 \ , \ u_1(x) > 0\}) = 0 ,$$

and this, together with (33), implies:

$$\sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^s u_1 D^r v \, dx = \langle q , v \rangle \qquad \forall v \in V_1 .$$

Putting then $\bar{\lambda} = -\frac{1}{p_0(x_1)}$, because $\mathbf{1} + \bar{\lambda} p_0 \in V_1$ and $\langle q , p_0 \rangle = 0$, from the previous relation follows:

$$\langle q , \mathbf{1} \rangle = 0$$

against the hypothesis. □

THEOREM 5. *In the* γ) *case, problem* (P) *allows solution iff, called* u_1 *the solution of* (24), *a real number* λ_1 *exists such that*:

$$(34) \qquad (u_1 - \lambda_1 p_0)^+ = 0 \qquad s\text{-a.e. } on \ \{x \in \Gamma_E \mid p_0(x) < 0\} .$$

When this condition occurs, $u_1 - \lambda_1 p_0$ *is solution of problem* (P).

Proof. About the necessity, given a solution u of problem (P), we let:

$$\lambda_1 = -\frac{u(x_1)}{p_0(x_1)} \quad , \quad u_1 = u + \lambda_1 p_0 ,$$

so that $u_1 \in V_1$. Observing that:

$$E_1 = E \quad \text{and} \quad u_1 = u \quad \text{on} \quad \{x \in \Gamma \mid p_0(x) = 0\}$$

and, from (14):

$$Eu^+ = 0 = E_1 u_1 \quad \text{on} \quad \{x \in \Gamma \mid p_0(x) < 0\} ,$$

we have, taking account of (6):

$$E_1 u_1^+ = Eu^+ \qquad s\text{-a.e. on } \Gamma$$

and consequently:

$$\int_\Gamma E_1 u_1^+ v \, ds = \int_\Gamma Eu^+ v \, ds \qquad \forall v \in W^2(\Omega) .$$

Hence, because u is solution of (3), u_1 is the solution of (24). Moreover, from (14) results:

$$(u_1 - \lambda_1 p_0)^+ = 0 \qquad s\text{-a.e. on} \quad \{x \in \Gamma_E \mid p_0(x) < 0\} .$$

Let us prove that the condition is sufficient. Let v be an element of $W^2(\Omega)$. Putting $\eta = -\dfrac{v(x_1)}{p_0(x_1)}$, because u_1 is solution of (24) and $v + \eta p_0 \in V_1$, results:

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs}\, D^s (u_1 - \lambda_1 p_0)\, D^r v \, dx + \int_\Gamma E_1 u_1^+ (v + \eta p_0)\, ds = \langle g , v \rangle . \tag{35}$$

On the other hand, from (34) results:

$$E_1 u_1^+ (v + \eta p_0) = E (u_1 - \lambda_1 p_0)^+ v \qquad s\text{-a.e. on} \quad \{x \in \Gamma \mid p_0(x) \le 0\} ,$$

and therefore, taking account of (6):

$$\int_\Gamma E_1 u_1^+ (v + \eta p_0)\, ds = \int_\Gamma E (u_1 - \lambda_1 p_0)^+ v \, ds . \tag{36}$$

From (35) and (36) follows that $u_1 - \lambda_1 p_0$ is solution of (3). □

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