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$\aleph_1$-Boolean spectrum, and stability

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Logica matematica. — $\aleph_1$*-Boolean spectrum and stability* (*). Nota di PIERO MANGANI (**) e ANNALISA MARCJA (***), presentata (****) dal Socio G. ZAPPA.

RIASSUNTO. — Si dimostra che la conoscenza delle algebre di Boole dei definibili di modelli di cardinità $\aleph_1$ di una teoria elementare è sufficiente per decidere il suo tipo di stabilità.

0. INTRODUCTION

In [1] and [2] we studied some properties of elementary theories, using Boolean algebras of parametrically definable subsets of their models. In this paper we will show that knowing such Boolean algebras of power $\aleph_1$ is sufficient to give information about stability of a theory (see Theorem 3.1).

In the following T will be a complete, quantifier eliminable theory in a countable language. $\mathcal{M}, \mathcal{M}', \mathcal{N}, \cdots$ will always denote models of T and the domains of these models will be denoted by $M, M', N, \cdots$. If $A \subseteq M$, the expanded structure $(\mathcal{M}, a)_{a \in A}$ will be denoted by $\mathcal{M}_A$ and its language by

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$L(A)$; F^1_A will be the set of formulas of $L(A)$ having only one free variable and by $\mathcal{B}(A)$ will be denoted the Boolean algebra obtained by F^1_A, modulo the equivalence relation: $\phi, \psi \in F^1_A, \phi \sim_A \psi$ if and only if $Th(\mathcal{M}_A) \models \forall v (\phi(v) \leftrightarrow \psi(v))$. The Stone space of $\mathcal{B}(A)$ will be denoted, as usual, by $S(A)$.

1. $\aleph_1$–Boolean spectrum of a theory

Let k be an infinite cardinal.

DEFINITION 1.1. *We call k-*Boolean spectrum *of* $T(\mathrm{Spec}_k(T))$ *the set of isomorphism types of* $\mathcal{B}(M)$, *where* $\mathcal{M} \models T$ *and* $|M| = k$.

Obviously $1 \leq |\mathrm{Spec}_k(T)| \leq 2^k$. The following theorem relates power of $\mathrm{Spec}_k(T)$ $(k \geq \aleph_1)$ with the categoricity of T.

THEOREM 1.2. *Let* $k \geq \aleph_1$. $|\mathrm{Spec}_k(T)| = 1$ *if and only if* T *is k-categorical.*

Proof.: $\leftarrow$ obvious.

$\rightarrow$ In [1] we proved the theorem (Thm. 3.7) under the hypotheses of ω-stability of T. But the hypothesis that $|\mathrm{Spec}_k(T)| = 1$ implies ω-stability of T, as remarked by D. Lascar and J. Baldwin (private communications). In fact it is easy to verify that types on $A, A \subseteq M$, realized by $\mathcal{M}$, correspond to equivalence classes on the set of atoms of $\mathcal{B}(M)$ $(At(\mathcal{B}(M))$, modulo the relation $m \equiv_A m_1$ if and only if for every $a \in \mathcal{B}(A)$, $m < a$ if and only if $m_1 < a$. As in [3] thm. 3.7 page 527, we can prove that for every infinite cardinal k, there exists $\mathcal{M} \models T$, $|M| = k$ such that for every $A \subseteq M$, $|A| \leq \aleph_0$, there exist at most countably many classes modulo A in $At(\mathcal{B}(M))$. If T is not ω-stable, there exists a model $\mathcal{N}$, $|N| = k$, having uncountably many classes $\equiv_N$ in $At(\mathcal{B}(N))$. It follows that $\mathcal{B}(M)$ cannot be isomorphic to $\mathcal{B}(N)$.

2. Trees in a Boolean algebra

DEFINITION 2.1. *A subset* $\mathcal{T}$ *of a Boolean algebra* $\mathcal{B}$ *is said a* tree *if the following conditions hold true*:

1) $0 \notin \mathcal{T}$, $1 \in \mathcal{T}$.

2) For every $x \in \mathcal{T}$ the set $\hat{x} = \{y \in \mathcal{T} : y <^* x\}$ is well ordered. ($<^*$ denotes the reverse order of $\mathcal{B}$).

The order type of x is said the *order* of x and is denoted by $o(x)$. A *branch* of $\mathcal{T}$ is a maximal linearly ordered subset of $\mathcal{T}$. The *length* of a branch X is defined as $\sup\{o(x) : x \in X\}$. *α-level* of $\mathcal{T}$ is the set U_α of the elements of $\mathcal{T}$, having order α.

DEFINITION 2.2. *If k is a cardinal and α an ordinal, $\mathscr{T}$ is (k, α)-tree if and only if every branch of $\mathscr{T}$ has length α and every element of $\mathscr{T}$ has exactly k pairwise disjoint immediate successors.*

If the theory T has a model $\mathscr{M}$ such that $\mathscr{B}(\mathrm{M})$ contains a tree $\mathscr{T}$, we shortly say that T *contains* $\mathscr{T}$.

Let $\mu_0 = \inf\{\mu \in \text{card} : 2^{\mu} > 2^{\aleph_0}\}$ and $k_0 = \inf\{k \in \text{card} : k > 2^{\aleph_0}$ and $k^{\aleph_0} > k\}$ (Observe that $\aleph_1 \leq \mu_0 \leq 2^{\aleph_0}$).

LEMMA 2.3. (a) T *contains a* $(2, \mu_0)$*-tree if and only if* T *contains a* $(2, \omega_1)$*-tree.* (b) T *contains a* (k_0, ω)*-tree if and only if* T *contains a* $(\aleph_1, \omega)$*-tree.*

Proof: (a) $\rightarrow$ trivial

$\leftarrow$ Remember that the language is countable; then there is a formula such that infinite instances of it occur in almost all branches. Hence the result follows by compactness theorem (details are omitted).

(b) $\rightarrow$ trivial

$\leftarrow$ In every α-level, infinite instance of the same formula ϕ_α occur. Again, the result follows by compactness theorem.

3. CONDITIONS FOR STABILITY

THEOREM 3.1. (a) T *is stable if and only if for every* $\mathscr{B} \in \mathrm{Spec}_{\aleph_1}(\mathrm{T})$, $\mathscr{B}$ *does not contain a* $(2, \omega_1)$*-tree.* (b) T *is superstable if and only if for every* $\mathscr{B} \in \mathrm{Spec}_{\aleph_1}(\mathrm{T})$, $\mathscr{B}$ *does not contain either a* $(2, \omega_1)$*-tree, or a* $(\aleph_1, \omega)$*-tree.* (c) T *is* ω*-stable if and only if for every* $\mathscr{B} \in \mathrm{Spec}_{\aleph_1}(\mathrm{T})$, $\mathscr{B}$ *does not contain a* $(2, \omega)$*-tree, if and only if every* $\mathscr{B} \in \mathrm{Spec}_{\aleph_1}(\mathrm{T})$ *is superatomic, if and only if every* $\mathscr{B} \in \mathrm{Spec}_{\aleph_0}(\mathrm{T})$ *is superatomic.*

Proof: (a) $\rightarrow$. If some $\mathscr{B} \in \mathrm{Spec}_{\aleph_1}(\mathrm{T})$ contains a $(2, \omega_1)$-tree, T contains a $(2, \mu_0)$-tree, by Lemma 2.3. Then the structure $\mathscr{A}$ generated by parameters occuring in the tree is such that $|\mathrm{A}| \leq 2^{\aleph_0}$ and $|\mathrm{S}(\mathrm{A})| = 2^{\mu_0} > 2^{\aleph_0}$, contradicting the stability of T (remember that T is stable if and only if it is $2^{\aleph_0}$ stable).

$\leftarrow$ If T is unstable, it follows that there is an unstable formula ϕ ([4] Thm. 2.13, page 36). By the " unstable formula theorem " ([4] Thm. 2.2, page 30) the set $\Gamma(\phi, \alpha)$ is consistent for every ordinal α and hence T contains a $(2, \omega_1)$-tree.

(b) $\rightarrow$ If T is superstable, then T does not contain a $(2, \omega_1)$-tree by (a); furthermore if T contains a $(\aleph_1, \omega)$-tree, T contains a (k_0, ω)-tree by lemma 2.3 (b). Then the structure A generated by parameters occuring in the tree is such that $|A| \leq k_0$ and $|S(A)| = k_0^{\aleph_0} > k_0$, contradicting the superstability of T (remember that T is superstable if and only if it is k_0-stable).

$\leftarrow$ It follows that T is stable. If T is not superstable, then T contains a $(\aleph_1, \omega)$-tree by Theorems 3.9 (page 46) and 3.14 (page 53) of Shelah [4].

(c) see [1], remembering that a Boolean algebra $\mathscr{B}$ is superatomic if and only if it does not contain any $(2, \omega)$-tree.

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