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Diffeomorphisms constructively associated with mutually diverging spacetimes which allow a natural identification of event points in general relativity. Part II

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Fisica matematica. — *Diffeomorphisms constructively associated with mutually diverging spacetimes which allow a natural identification of event points in general relativity.* Part II. Nota di GAETANO ZAMPIERI (*), presentata (**) dal Corrisp. A. BRESSAN.

RIASSUNTO. — In questo lavoro si dà una definizione di divergenza fra cronotopi della Relatività Generale e si costruisce un criterio per l'identificazione dei punti eventi di cronotopi divergenti che appartengono ad una classe consistente con la presenza di campi elettromagnetici nel vuoto.

4. INTRODUCTION (1).

In Sect. 5 the diffeomorphisms that provide the quasi-absolute concept of event point are constructed. To be more detailed, let us consider the typical equivalence class $\mathscr{D}$ of the divergence relation, assume that $S_4^1, S_4^2 \in \mathscr{D}$, and that P^{21} be a region of type $\mathscr{P}$, with respect to both spacetimes, where the structures of S_4^1 and S_4^2 coincide. Suppose $\mathscr{E}^1 \in S_4^1$ and consider the maximal integral line c_1 of the reference frame η^1 on S_4^1, starting from $\mathscr{E}^1$ (2) (whose parameter is the proper time (3)).

On the basis of the definition of a type $\mathscr{P}$ region, there is a value $-\tau$ of the parameter, such that $\tau \geq 0$ and $\mathscr{E} = c_1(-\tau) \in P^{21}$.

Now consider the maximal integral line c_2, of the reference frame η^2 on S_4^2, starting from $\mathscr{E}$. If its parameter takes on the value τ, then the event $\mathscr{E}^2 = c_2(\tau)$ is by definition the correspondent of $\mathscr{E}^1$ in S_4^2. Otherwise, $\mathscr{E}^1$ has no correspondent.

This definition makes sense because $\mathscr{E}^2$, or its inexistence, is independent of the choice of P^{21} and τ.

Thus a bijection f^{21}, from a subset of S_4^1 onto a subset of S_4^2, is defined. It turns out to be a diffeomorphism whose restriction to P^{21} is the inclusion map. Moreover, f^{21} is the identity map if the two spacetimes coincide, and $f^{21} = (f^{12})^{-1}$, where f^{12} is the diffeomorphism associated with the inverse

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(1) This section is the sequel of Sect. 1 in Part. I—see ref. [7].

(2) We have (*i*) $c_1 : \mathfrak{I}(\mathscr{E}^1) \to S_4^1$. where $\mathfrak{I}(\mathscr{E}^1)$ is an open interval of $\mathbf{R}$ containing zero, (*ii*) $c_1(0) = \mathscr{E}^1$, (*iii*) $\dot{c}_1(\tau) = \eta^1[c_1(\tau)]\ \forall \tau \in \mathfrak{I}(\mathscr{E}^1)$, where $\dot{c}_1(\tau)$ is the tangent vector at $c_1(\tau)$, and (*iv*) c_1 is maximal—see e.g. ref [5] p. 4.

(3) See e.g. ref. [5] p. 41.

pair (S_4^2, S_4^1). Finally, if $\mathscr{E}^1$ has a correspondent $\mathscr{E}^2$, and $\mathscr{E}^2$ has a correspondent $\mathscr{E}^3$ in a third spacetime $S_4^3 \in \mathscr{D}$, then $\mathscr{E}^3$ corresponds to $\mathscr{E}^1$ under the diffeomorphism f^{31} associated with (S_4^1, S_4^3) (4).

The section ends with the definition of a second class of spacetimes, $\mathfrak{B}$, whose typical element S_4 belongs to $\mathfrak{A}$, and is such that the reference frame η is a complete vector field (5).

In this case the diffeomorphism f^{21} maps the first spacetime onto the second. Thus in the class $\mathfrak{B}$ we have an absolute concept of event point.

5. Diffeomorphisms associated with mutually diverging spacetimes

Let S_4^k $(k = 1, 2)$ be spacetimes in $\mathfrak{A}$—see ref. [7] Def. 3—and mutually diverging—see ref. [7] Def. 2. Let P^{21} be as in ref. [7] Def. 2, and let η^k $(k = 1, 2)$ be the analogue for S_4^k of the reference frame η on S_4 mentioned in [7] Def. 3.

Lemma 2. *Under these assumptions*

$$\eta^1 |_{P^{21}} = \eta^2 |_{P^{21}} .$$

Proof. If $\mathscr{E} \in P^{21}$ and $k \in \{1, 2\}$, then, by Def. 1 in [7], the causal past of $\mathscr{E}$ in S_4^k coincides with the one in P^{21} endowed with the structure induced from the structure of S_4^k (6). Furthermore, the structures of S_4^1 and S_4^2 coincide on P^{21}—see [7] Def. 2. This, and the definition of $\eta^k(\mathscr{E})$—which depends only on the causal past of $\mathscr{E}$—, yield the proposition. q.e.d.

Let $\mathscr{E}^1 \in S_4^1$ and let $c_1 : \mathfrak{J}(\mathscr{E}^1) \to S_4^1$ be the maximal integral line of η^1 starting from $\mathscr{E}^1$ (i.e. with $c_1(0) = \mathscr{E}^1$)—see fnt (2). Then the map $\bar{c} : \{\alpha \in \mathbf{R} : \alpha \geq 0, -\alpha \in \mathfrak{J}(\mathscr{E}^1)\} \to S_4^1, \alpha \mapsto c_1(-\alpha)$ is a causal past-pointing and past-inextendible curve (7); and it enters P^{21} and then remains within it because $P^{21} \in \mathscr{P}(S_4^1)$—see Def. 1 in [7]. Therefore we can consider $\tau \in \mathbf{R}$ with $\tau \geq 0, -\tau \in \mathfrak{J}(\mathscr{E}^1)$, and $c_1(-\tau) \in P^{21}$. Furthermore,

$$\tau' \geq \tau, -\tau' \in \mathfrak{J}(\mathscr{E}^1) \Rightarrow c_1(-\tau') \in P^{21}. \tag{1}$$

(4) The event $\mathscr{E}^1$ can have a correspondent $\mathscr{E}^3$ in S_4^3 without any correspondent in S_4^2.

(5) That is the domain of all its maximal integral curves is the whole $\mathbf{R}$.

(6) In fact the causal past—see [4] p. 183—of $\mathscr{E}$ coincides with the union (of the ranges) of the past-pointing and past-inextendible causal curves starting from $\mathscr{E}$.

(7) $\bar{c}$ is past-inextendible, i.e. it has no past-endpoint—see fnt 7 in Part I i.e. [7]—because it is a maximal integral curve of a causal vector field (which is nowhere vanishing).

Now let $\mathscr{E} = c_1(-\tau)$ and let $c_2 : \mathfrak{J}(\mathscr{E}) \to S_4^2$ be the maximal integral curve of the vector field η^2 starting from $\mathscr{E}$ (i.e. with $c_2(0) = \mathscr{E}$).

If $\tau \in \mathfrak{J}(\mathscr{E})$, then I say that $\mathscr{E}^1$ is R^{21}—related to $\mathscr{E}^2 = c_2(\tau)$.

This definition is good because $\mathscr{E}^2$, or its inexistence, is independent of the aforementioned choice of P^{21} and τ. In fact, let $\overline{P}^{21}$ and $\overline{\tau}$ satisfy the same conditions, and let us suppose that $\overline{\tau} > \tau$. Then (1) yields $c_1(-\tau') \in P^{21}$ for any $\tau' \in [\tau, \overline{\tau}]$. Now, using Lemma 2, we have $c_2(\alpha) = c_1(\alpha - \tau)$ for any $\alpha \in [\tau - \overline{\tau}, 0]$. Furthermore, if $\overline{c}_2 : \mathfrak{J}(\overline{\mathscr{E}}) \to S_4^2$ is the maximal integral line of the vector field η^2 with $\overline{c}_2(0) = \overline{\mathscr{E}} = c_1(-\overline{\tau})$, then $\overline{c}_2(\alpha) = c_1(\alpha - \overline{\tau})$ for any $\alpha \in [0, \overline{\tau} - \tau]$. Therefore $\overline{c}_2$ coincides with

$$\{\alpha \in \mathbf{R} : (\alpha + \tau - \overline{\tau}) \in \mathfrak{J}(\mathscr{E})\} \to S_4^2, \ \alpha \mapsto c_2(\alpha + \tau - \overline{\tau}).$$

This yields (*i*) $\tau \in \mathfrak{J}(\mathscr{E})$ if and only if $\overline{\tau} \in \mathfrak{J}(\overline{\mathscr{E}})$, and (*ii*) $\overline{c}_2(\overline{\tau}) = c_2(\tau) = \mathscr{E}^2$ for $\tau \in \mathfrak{J}(\mathscr{E})$.

This completes the proof because the case where $\overline{\tau} = \tau$ is trivial, and the one where $\overline{\tau} < \tau$ is similar (it is enough to use the analogue of (1) and Lemma 2 with $\overline{\tau}$ instead of τ and $\overline{P}^{21}$ instead of P^{21}).

We have

$$(\mathscr{E}^1, \mathscr{E}^2), (\tilde{\mathscr{E}}^1, \mathscr{E}^2), (\mathscr{E}^1, \tilde{\mathscr{E}}^2) \in R^{21} \Rightarrow \tilde{\mathscr{E}}^1 = \mathscr{E}^1, \tilde{\mathscr{E}}^2 = \mathscr{E}^2.$$

Therefore R^{21} is the graph of a bijection $f^{21} : Dom\, f^{21} \to Ran\, f^{21}$ with

(2) $$(Dom\, f^{21}) \cap (Ran\, f^{21}) \supset P^{21} \qquad \text{and} \quad f^{21}|_{P^{21}} = i_{P^{21}},$$

where $i_{P^{21}} : P^{21} \to (Ran\, f^{21})$ is the inclusion map.

If S_4^2 coincides with S_4^1, then we can write f^{11} instead of f^{21} and we have

(3) $$f^{11} = i_{S_4^1} \quad \text{— identity map.}$$

In this argument I have used the reflexivity of the divergence relation. Its symmetry allows the exchange of the roles of S_4^1 and S_4^2. This gives a map f^{12} which satisfies

(4) $$f^{12} = (f^{21})^{-1}.$$

Now, let us use the transitivity, and let us consider S_4^3 in the same equivalence class as S_4^1 and S_4^2. If R^{32} [R^{31}] is the analogue for S_4^2 [S_4^1] and S_4^3 of the relation R^{21} on S_4^1 to S_4^2, then

$$(\mathscr{E}^1, \mathscr{E}^2) \in R^{21}, (\mathscr{E}^2, \mathscr{E}^3) \in R^{32} \Rightarrow (\mathscr{E}^1, \mathscr{E}^3) \in R^{31}.$$

But $\mathscr{E}^1$ can be R^{31}—related to $\mathscr{E}^3$ also in the case where $(\mathscr{E}^1, \mathscr{E}^2) \notin R^{21}$ and $(\mathscr{E}^2, \mathscr{E}^3) \notin R^{32}$. Thus

$$f^{32} \circ f^{21} : \{\mathscr{E}^1 \in Dom\, f^{21} : f^{21}(\mathscr{E}^1) \in Dom\, f^{32}\} \to Ran\, f^{32}$$

is generally different from f^{31}. However

(5) $\mathscr{E}^1 \in Dom\, f^{21}, f^{21}(\mathscr{E}^1) \in Dom\, f^{32} \Rightarrow \mathscr{E}^1 \in Dom\, f^{31}, (f^{32} \circ f^{21})(\mathscr{E}^1) = f^{31}(\mathscr{E}^1)$.

Lastly

PROPOSITION 3. *The map* f^{21} *is a* C^∞ *diffeomorphism.*

Proof. If f^{21} is proved to be C^∞, then f^{12} is C^∞ too, and (4) gives the thesis.

For $k = 1, 2$, assume that (*i*) $\mathscr{E}^k \in S_4^k$, (*ii*) $c_{\mathscr{E}^k} : \mathfrak{J}(\mathscr{E}^k) \to S_4^k$ is the maximal integral curve of the vector field η^k starting from $\mathscr{E}^k$, (*iii*) $\tau \in \mathbf{R}$, then we set

$$E_\tau^k = \{\mathscr{E}^k \in S_4^k : \tau \in \mathfrak{J}(\mathscr{E}^k)\} \qquad \text{and} \quad \mu_\tau^k : E_\tau^k \to S_4^k, \mathscr{E}^k \mapsto c_{\mathscr{E}^k}(\tau).$$

(Usually the set $\{\mu_\tau^k : \tau \in \mathbf{R}\}$ is called the flow of the vector field η^k). Since η^k is C^∞, then E_τ^k is open and μ_τ^k is C^∞ ($\forall\, \tau \in \mathbf{R}$). By definition, μ_τ^k has a C^∞ inverse which obviously coincides with $\mu_{-\tau}^k$.

Let P^{21} be as in [7] Def. 2, and let us fix (arbitrarily) an event $\mathscr{E}^1 \in Dom\, f^{21}$. Moreover let $\tau \in \mathbf{R}$ be such that $\tau \geq 0$ and $c_{\mathscr{E}^1}(-\tau) \in P^{21}$.

Since (*i*) E_τ^1 and P^{21} are open in S_4^1. (*ii*) E_τ^2 is open in S_4^2, and (*iii*) the induced topologies of S_4^1 and S_4^2 coincide on P^{21}—see [7] Def. 2—then $P^{21} \cap E_\tau^1 \cap E_\tau^2$ is open, and so is its image $U = \mu_\tau^1(P^{21} \cap E_\tau^1 \cap E_\tau^2)$ underf μ_τ^1, an homeomorphism into.

Now, let us observe that $\mathscr{E}^1 \in U, U \subset Dom\, f^{21}$, and $f^{21}|_U = (\mu_\tau^2 \circ \mu_{-\tau}^1)|_U$. This implies that f^{21} is C^∞ in the neighbourhood U of the (arbitrary) point $\mathscr{E}^1$, because $\mu_{-\tau}^1$ and μ_τ^2 are C^∞. q.e.d.

* * *

It is interesting to consider the following class $\mathfrak{B}$ of spacetimes.

DEFINITION 4. The spacetime S_4 is said to belong to $\mathfrak{B}$ if $S_4 \in \mathfrak{A}$—see Def. 3 in ref. [7]—and the reference frame η is a complete vector field—see footnote (5).

The use of $\mathfrak{B}$ instead of $\mathfrak{A}$ in the preceding arguments yields that, in connection with $\mathfrak{B}$, f^{21} is a C^∞ diffeomorphism from S_4^1 onto S_4^2, and (5) becomes

$$f^{31} = f^{32} \circ f^{21}.$$

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