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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Some Characterization of the q–Gamma Function by Functional Equations.* Nota II di MARINO BADIALE, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questo lavoro, suddiviso in una Nota I e in una Nota II, si estendono alle funzioni q–gamma i classici risultati sulla determinazione univoca della funzione gamma tramite equazioni funzionali; si introduce poi una q–generalizzazione di una funzione fattoriale intera, e se ne indicano le principali proprietà.

2 The counterexamples which conclude part I serve to indicate the obstacles to a reasonable extension of theorem b) to the functions $\Gamma_q(x)$ It is, however, possible to weaken the assumption that d^2f/dx^2 be continuous, and that (1.2) holds for all q, if one strengthens the remaining conditions. More precisely, we have:

PROPOSITION 2. *Let $f(q, x)$ be a real valued function for $q > 0$ $x > 0$ such that df/dx exists for all (q, x). Suppose that $f(q, x)$ satisfies (1.1) and that there exists a $q_0 \neq 1$ such that $f(q_0, x) > 0$ for all x and*

$$(2.1)\qquad f(q_0, nx) f(q_0^n, 1/n), \cdots, f(q_0^n, (n-1)/n) =$$
$$= f(q_0^n, x) f(q_0^n, x + 1/n), \cdots, f(q_0^n, x + (n-1)/n) (1 + q_0 + \cdots + q_0^{n-1})^{nx-1}$$

for all $x > 0$ and arbitrarily large positive integers n. Let $\varphi(q, x) = f(q, x)/\Gamma_q(x)$ and $g(q, x) = \log \varphi(q, x)$. Suppose that the sequence $g_x(q_0^n, x)$ converges uniformly in x as $n \to \infty$ to a function $h(x)$ integrable on $0 \leq x \leq 1$, and that the sequence $g(q_0^n, x)$ converges for at least one value of x, $0 < x < 1$.

(*) Nella seduta dell'8 gennaio 1983.

Then $f(q_0, x)$ differs from $\Gamma_q(x)$ by at most a multiplicative constant, that is $f(q_0, x) \equiv k\Gamma_{q_0}(x)$ for some constant k.

Proof. As in the proof of proposition 1, $\varphi(q, x)$ is periodic in x with period 1 and so we need only consider x such that $0 \leq x \leq 1$ Replacing x by x/n in (2.1) and passing to $\varphi(q, x)$ gives

$$\varphi(q_0, x)\,\varphi(q_0^n, 1/n), \cdots, \varphi(q_0^n, (n-1)/n) = \tag{2.2}$$

$$= \varphi(q_0^n, x/n)\,\varphi(q_0^n, (x+1)/n), \cdots, \varphi(q_0^n, (x+n-1)/n).$$

Taking the logarithmic derivative of both sides gives

$$g_x(q_0, x) = \frac{1}{n}\,[g_x(q_0^n, x/n) + \cdots + g_x(q_0^n . (x+n-1)/n)]. \tag{2.3}$$

Hence we find

$$g_x(q_0, x) = \frac{1}{n} \sum_{k=1}^{n-1} g(q_0, (x+k)/n) - h((x+k)/)n + \frac{1}{n} \sum_{k=1}^{n-1} h((x+k)/n).$$

By the assumed uniform convergence of $g_x(q_0^n, x)$ to $h(x)$ the first sum here tends to zeıo as $n \to \infty$, while the second sum, being a Riemann sum for $h(x)$ on the interval $0 \leq x \leq 1$, has limit $\int_0^1 h(x)\,dx$. On the other hand, by the hypotheses made on the sequence $g_x(q_0^n, x)$ we find that $g(q_0^n, x)$ converges uniformly as $n \to \infty$ to a function $H(x)$ such that $d/dx\,(H(x)) = h(x)$. But then it is clear that $\int_0^1 h(x)\,dx = H(1) - H(0) = 0$, since $H(x)$ has period 1, being a uniform limit of functions of period 1. Thus $g_x(q_0, x) = 0$, and so $g(q_0, x)$ is constant, and the same holds for $\varphi(q_0, x)$. QED.

The counterexample preceding this proposition satisfies all the conditions except (2.1), when q_0 is taken less than 1 (greater than 1 if the exponent in $h_q(x)$ is -4). Observe that it is not assumed that df/dx is continuous, although, of course, this hypothesis " after being thrown out of the door, has returned through the window " in the guise of our convergence assumption. As in the corollary to proposition 1 we obtain a good analogue to theorem b) if we consider the domain $0 \leq q \leq 1, 0 < x$:

COROLLARY: *let $f(q, x)$ be a positive, continuous, real valued function for $0 \leq q \leq 1, 0 < x$ such that df/dx is continuous.*

Suppose that $f(q, x)$ satisfies (1.1) and (2.1) for some positive integer n and all $q < 1$. Then $f(q, x) = k_q\,\Gamma_q(x)$ for some constant k_q, depending on q.

Proof. Iteration shows that if (2.1) holds for n and all q, then it holds for n^2, n^4 etc., and hence for arbitrarily large values. The function $g_x(q, x)$ is then uniformly continuous on $0 \leq q \leq 1, 0 \leq x \leq 1$, and this implies uniform convergence of the $g_x(q^n, x)$ to $g_x(0, x)$ as $n \to \infty$ through the ' good ' values. Clearly, $g(q^n, x)$ converges to $g(0, x)$ for all x and all $q < 1$. QED.

3. We now seek to extend theorem c) to the functions $\Gamma_q(x)$. We restrict ourselves to the case $0 \leq q \leq 1$. It turns out that in order to recover a good analogue of theorem c) it is sufficient to impose a rather weak additional hypothesis, the existence of a continuous derivative with respect to the variable q.

PROPOSITION 3. *Let $f(q, x)$ be a positive real-valued continuous function on $0 \leq q \leq 1, 0 < x$ such that df/dq is continuous. Suppose that $f(q, x)$ satisfies (1.1) and (2.1) for all (q, x) and all positive integers n. Then $f(q, x) = \Gamma_q(x)$.*

Proof. We use the notation of proposition 2; (2.1) now holds for all (q, x) and all n. Taking logarithmic derivatives with respect to q gives that $h(q, x) = d/dq(\log \varphi(q, x))$ satisfies $h(q, x) + nq^{n-1} h(q^n, 1/n) + \cdots + nq^{n-1} \times \times h(q^n, (n-1)/n) = nq^{n-1}(h(q^n, x/n) + \cdots + h(q^n, (x+n-1)/n))$.

Adding and subtracting $h(q^n, 0)$ and rearranging we find

$$h(q, x) = nq^{n-1}\left[h(q^n, 0) + \sum_{k=0}^{n-1} h(q^n, (x+k)/n - h(q^n, k/n)\right]. \tag{3.1}$$

However, it is not difficult to show that the right hand side of (3.1) tends to 0 for $q < 1$ as $n \to \infty$. Indeed we have, on multiplying and dividing by n,

$$h(q, x) = n^2 q^{n-1}\left[\frac{1}{n} h(q^n, 0) + n^{-1} \sum_{k=0}^{n-1} h(q^n, (x+k)/n) - h(q^n, k/n)\right]$$

and

$$n^{-1} \sum_{k=0}^{n-1} h(q^n, (x+k)/n) - h(q^n, k/n) = n^{-1} \sum_{k=0}^{n-1} (h(q^n, (x+k)/n) -$$

$$- h(0, (x+k)/n)) + n^{-1} \sum_{k=0}^{n-1} (h(0, (x+k)/n) - h(0, k/n) +$$

$$+ n^{-1} \sum_{k=0}^{n-1} (h(0, k/n) - h(q^n, k/n)).$$

By the uniform continuity of $h(q, x)$ on the square $0 \leq q \leq 1, 0 \leq x \leq 1$ all three terms tend to 0 as $n \to \infty$.

We conclude that $h(q, x) = 0$ on the closed square (the case $q = 1$ $0 \leq x \leq 1$ is already covered by Artin's theorem c) and our conventional interpretation of the functional equation for $q = 1$, or else, follows by continuity of $h(q, x)$).

Thus $g(q, x) = \log (f(q, x)/\Gamma_q(x))$ is independent of q and so we may write $g(q, x) = g(x)$ and (2.1) gives

$$g(nx) + g(1/n) + \cdots + g((n-1)/n) = \tag{3.2}$$
$$= g(x) + g(x + 1/n) + \cdots + g(x + (n-1)/n).$$

At this point we may follow Artin's path. Let $g(x)$ have Fourier series

$$g(x) \sim \sum_{\nu=-\infty}^{+\infty} c_\nu \, e^{2\pi i \nu n x}.$$

Then, the Fourier series of the left hand side of (3.2) is given by $\sum_{\nu=-\infty}^{+\infty} \mathrm{d}_\nu \, e^{2\pi i \nu x}$ with $\mathrm{d}_\nu = c_\nu$ for $\nu \neq 0$ and $\mathrm{d}_0 = c_0 + g(1/n) + \cdots + g((n-1)/n)$.

The right hand side has Fourier series given by

$$\sum_{k=0}^{n-1} \sum_{\nu=-\infty}^{+\infty} c_\nu \, e^{2\pi i \nu x} \, e^{2\pi i \nu k/n} = \sum_{\nu=-\infty}^{+\infty} c_\nu \left(\sum_{k=0}^{n-1} e^{2\pi i \nu k/n} \right) e^{2\pi i \nu x},$$

which by the usual relation

$$\sum_{k=0}^{n-1} e^{2\pi i \nu k/n} = \begin{cases} n & \text{if } n \mid \nu \\ 0 & \text{otherwise} \end{cases}$$

becomes $\sum_{\nu=-\infty}^{+\infty} n c_{n\nu} \, e^{2\pi i \nu n x}$.

Thus we have

$$\sum_{\nu=-\infty}^{+\infty} \mathrm{d}_\nu \, e^{2\pi i \nu n x} = \sum_{\nu=-\infty}^{+\infty} n c_{n\nu} \, e^{2 i \pi \nu n x} \qquad \text{and so} \quad \mathrm{d}_\nu = n c_{p\nu}.$$

Hence we obtain that for $\nu \neq 0$ $c_\nu = n c_{n\nu}$, that is, in particular:

$$c_n = c_1/n \qquad \text{and} \quad c_{-n} = c_{-1}/n \qquad \text{for all integers} \quad n > 0. \tag{3.3}$$

If we now replace $g(x)$ by $g(x) - c_0$, the new function satisfies the conditions (3.3) and has constant term in its Fourier series equal to 0. As in Artin's proof, this now gives $g(x) = c_0$. But then $\varphi(q, x)$ is also constant, and since $\varphi(1, x) = 1$, we conclude that $f(q, x) = \Gamma_q(x)$ as desired. QED.

4. We conclude our discussion with some remarks relating to $\Gamma_q(x)$ considered as an analytic function of its arguments. As in the case of the usual gamma function, analyticity in z (or even in z and q) together with the functional equation (1.1) does not characterize $\Gamma_q(z)$ uniquely. In fact, if we multiply by any analytic function periodic with period 1 we obtain another function, satisfying (1.1), and if we demand that our multiplier assume the value 1 at all integers (as does, for example, $\cos 2\pi z$), the new function will interpolate $n!_q$. This technique will always lead to meromorphic functions, like $\Gamma_q(z)$ itself.

We can, however, search for an entire function which interpolates $n!_q$. In other words we seek a q-analogue of the following function, introduced by Hadamard:

$$H(z) = (\Gamma(1-z))^{-1}\, d/dz\, [\log [\Gamma((1-z)/2) \big/ \Gamma(1-(z/2))]] .$$

$H(z)$ interpolates $n!$, is entire, and satisfies the functional equation

$$H(z+1) = zH(z) + (\Gamma(1-z))^{-1}.$$

In this regard we have the following.

PROPOSITION 4: *Define, for* $q > 0$

$$H_q(z) = kq^{\binom{z}{2}}\, \Gamma_q(1-z)^{-1}\, d/dz\, [\log [\Gamma_q((1-z)/2) \big/ \Gamma_q(1-z/2)]]$$

with $k = (q-1)/\log q$.

Then $H_q(z)$ is an entire function of z which interpolates $n!_q$ and which satisfies the functional equation

$$H_q(z+1) = \frac{1-q^z}{1-q}\, H_q(z) + \tfrac{1}{2}\, q^{\binom{z}{2}}\, (1 + q^{z/2})\, (\Gamma_q(1-z))^{-1}. \tag{4.1}$$

Proof. It is easy to verify that $\Gamma_q(z)$ has (simple) poles at the points $x = -n + 2k\pi i/\log q$ for $n = 0, 1, 2, \cdots$ and $k = 0, \pm 1, \pm 2, \cdots$, and has no zeroes. Furthermore, the logarithmic derivative appearing in the definition has poles (simple, of course) in precisely the points where $1/\Gamma_q(1-z)$ vanishes, namely the points of the form $n + 2k\pi i/\log q$ with $n = 1, 2, 3, \cdots$ and $k = 0, \pm 1, \pm 2, \cdots$. In fact, the numerator contributes the poles 'over' the odd integers, while the denominator contributes the poles with n an even integer. Hence $H_q(z)$ is entire. In particular $H_q(0)$ is finite, so if (4.1) holds we will have $H_q(1) = 1$, and by recursion $H_q(n+1) = n!_q = \Gamma_q(n+1)$ for $n = 0, 1, 2, \cdots$. Thus it remains only to establish (4.1).

By definition we have

$$H_q(z+1) = kq^{(z^2+z)/2}\, (\Gamma_q(-z))^{-1}\, d/dx\, [\log [\Gamma_q(-z/2) \big/ \Gamma_q((1-z)/2)]] =$$

$$- kq^{(z^2+z)/2}\, (\Gamma_q(-z))^{-1}\, d/dx\, [\log [\Gamma_q((1-z)/2) \big/ \Gamma_q(-z)/2)]] .$$

It follows from (1.1) that

$$\Gamma_q(-z) = \frac{1-q}{1-q^{-z}}\, \Gamma_q(1-z)$$

and

$$\Gamma_q(-z/2) = \frac{1-q}{1-q^{-z/2}}\, \Gamma_q(1-z/2) .$$

Hence

$$\mathrm{H}_q(z+1) = -kq^{(z^2+z)/2}\,\Gamma_q(1-z)\,\frac{1-q^{-z}}{1-q}\,\mathrm{d}/\mathrm{d}x\left[\log\frac{\Gamma_q((1-z)/2)}{\Gamma_q(1-z/2)}\right] -$$

$$-kq^{(z^2+z)/2}\,(\Gamma_q(1-z))^{-1}\,\frac{1-q^{-z}}{1-q}\,\mathrm{d}/\mathrm{d}x\left[\log\frac{1-q^{-z/2}}{1-q}\right] =$$

$$= kq^{(z^2-z)/2}\,(\Gamma_q(1-z))^{-1}\,\frac{1-q^{z}}{1-q}\,\mathrm{d}/\mathrm{d}x\left[\log\frac{\Gamma_q((1-z)/2]}{\Gamma_q(1-z/2)}\right] +$$

$$+ kq^{(z^2-z)/2}\,(\Gamma_q(1-z))^{-1}\,\frac{1-q^{z}}{1-q}\,\tfrac{1}{2}\log q\cdot\frac{1}{q^{z/2}-1} =$$

$$= \frac{1-q^z}{1-q}\,\mathrm{H}_q(z) + \tfrac{1}{2}\,\frac{1-q^{z/2}}{1-q^z}\,(\Gamma_q(1-z))^{-1}\,q^{(z^2-z)/2} =$$

$$= \frac{1-q^z}{1-q}\,\mathrm{H}_q(z) + \tfrac{1}{2}\,(1+q^{z/2})\,(\Gamma_q(1-z))^{-1}\,q^{(z^2-z)/2}. \qquad \text{QED.}$$

Needless to say, when $q \to 1$ then $\mathrm{H}_q(z) \to \mathrm{H}(z)$ and the functional equation (4.1) tends to the equation of $\mathrm{H}(z)$.

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