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Limits of minimum problems for general integral functionals with unilateral obstacles

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Calcolo delle variazioni. — *Limits of minimum problems for general integral functionals with unilateral obstacles* (*). Nota di GIANNI DAL MASO, presentata (**) dal Corrisp. E. DE GIORGI.

RIASSUNTO. — Se il problema di minimo ($\mathscr{P}_\infty$) è il limite, in senso variazionale, di una successione di problemi di minimo con ostacoli del tipo

$$(\mathscr{P}_h) \qquad \min_{u \geq \psi_h} \int_A [f_h(x, u, Du) + b(x, u)]\, dx,$$

allora ($\mathscr{P}_\infty$) può essere scritto nella forma

$$(\mathscr{P}_\infty) \qquad \min_u \left\{ \int_A [f_\infty(x, u, Du) + b(x, u)]\, dx + \int_A g_\infty(x, \tilde{u}(x))\, d\lambda_\infty(x) \right\} \cdot$$

dove $\tilde{u}$ è un conveniente rappresentante di u e λ_∞ è una misura non negativa.

INTRODUCTION

In this paper we are concerned with sequences of minimum problems with obstacles of the form

$$(\mathscr{P}_h) \qquad \min_{u \geq \psi_h} \int_A [f_h(x, u(x), Du(x)) + b(x, u(x))]\, dx .$$

Under suitable hypotheses we show that, if there exists a " limit problem " ($\mathscr{P}$), then ($\mathscr{P}$) can be written in the form

$$(\mathscr{P}) \qquad \min_u \left\{ \int_A [f(x, u(x), Du(x)) + b(x, u(x))]\, dx + \int_A g(x, \tilde{u}(x))\, d\lambda(x) \right\},$$

where λ is a non-negative measure and $\tilde{u}(x) = \frac{1}{2}[\tilde{u}_+(x) + \tilde{u}_-(x)]$, with

$$\tilde{u}_+(x) = \limsup_{\varepsilon \to 0+} (\varepsilon\pi)^{-n/2} \int_{\mathbf{R}^n} u(y) \exp\left[-\frac{(x-y)^2}{\varepsilon}\right] dy,$$

$$\tilde{u}_-(x) = \liminf_{\varepsilon \to 0+} (\varepsilon\pi)^{-n/2} \int_{\mathbf{R}^n} u(y) \exp\left[-\frac{(x-y)^2}{\varepsilon}\right] dy .$$

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(**) Nella seduta del 12 febbraio 1983.

More precisely we prove that, for every choice of the function $b(x, s)$ in a suitable class of admissible functions, the sequences of the minimum points and of the minimum values of the problems $(\mathscr{P}_h)$ converge, as $h \to +\infty$, to the minimum point and to the minimum value of the problem $(\mathscr{P})$.

The function f does not depend on the obstacles ψ_h, whereas the function g and the measure λ depend both on the obstacles ψ_h and on the integrands f_h.

The first examples where the functional

$$G(u, A) = \int_A g(x, \tilde{u}(x))\, d\lambda(x)$$

takes every positive real value were proved by L. Carbone and F. Colombini [4].

This paper extends the results obtained by E. De Giorgi, G. Dal Maso, P. Longo [10], G. Dal Maso, P. Longo [8], H. Attouch, C. Picard [1], D. Cioranescu, F. Murat [5], G. Dal Maso [6], with a considerable improvement: the integrands $f_h(x, s, z)$ are not supposed to be convex in s, thus the functional

$$G(u, A) = \int_A g(x, \tilde{u}(x))\, d\lambda(x)$$

can be non-convex (see example 1). This leads to a deep change in the proofs, which are completely different from the proofs of the quoted papers.

1. The Main results

Throughout this paper p is a real number, with $1 \leq p < +\infty$, and n is an integer, with $n \geq 1$.

Let X be a lattice. We say that a function $F : X \to [0, +\infty]$ is *sub-modular* if

$$F(x \wedge y) + F(x \vee y) \leq F(x) + F(y)$$

for every $x, y \in X$.

We denote by $\mathscr{A}$ the family of all bounded open subsets of $\mathbf{R}^n$. We say that a functional $F : L^p(\mathbf{R}^n) \times \mathscr{A} \to [0, +\infty]$ is *local* if $F(u, A) = F(v, A)$ for every $A \in \mathscr{A}$ and for every pair of functions $u, v \in L^p(\mathbf{R}^n)$ such that $u = v$ a.e. on A.

We say that a local functional F is *sub-modular* (resp. *decreasing*, *convex*, etc.) if, for every $A \in \mathscr{A}$, the function $u \mapsto F(u, A)$ is sub-modular (resp. decreasing, convex, etc.) on $L^p(\mathbf{R}^n)$.

We say that the local functional F is a *measure* if, for every $u \in L^p(\mathbf{R}^n)$, the set function $A \mapsto F(u, A)$ is the trace on $\mathscr{A}$ of a countably additive set function defined on the Borel σ-field of $\mathbf{R}^n$ and with values in $[0, +\infty]$.

Let Φ_p be the local functional on $L^p(\mathbf{R}^n)$ defined by

$$\Phi_p(u, A) = \begin{cases} \int_A |Du|^p \, dx & \text{if } u \in L^p(\mathbf{R}^n) \cap W^{1,p}(A), \\ +\infty & \text{if } u \in L^p(\mathbf{R}^n) - W^{1,p}(A), \end{cases}$$

and let $\overline{\Phi}_p$ be the local functional on $L^p(\mathbf{R}^n)$ defined by $\overline{\Phi}_p = \Phi_p$, if $1 < p < +\infty$, and by

$$\overline{\Phi}_1(u, A) = \sup \left\{ \int_A u \operatorname{div} \varphi \, dx : \varphi \in C_0^1(A, \mathbf{R}^n), |\varphi| \leq 1 \right\},$$

if $p = 1$.

Let $c \geq 1$ be a constant. We denote by $\mathcal{F} = \mathcal{F}(c)$ the class of all *sub-modular local* functionals F on $L^p(\mathbf{R}^n)$ which are *measures* such that

$$\overline{\Phi}_p(u, A) \leq F(u, A) \leq c \left[\Phi_p(u, A) + \int_A |u|^p \, dx + \operatorname{meas}(A) \right]$$

for every $u \in L^p(\mathbf{R}^n)$ and for every $A \in \mathcal{A}$.

Examples of functions of the class $\mathcal{F}$ are the integral functionals

$$F(u, A) = \begin{cases} \int_A f(x, u(x), Du(x)) \, dx & \text{if } u \in L^p(\mathbf{R}^n) \cap W^{1,p}(A), \\ +\infty & \text{if } u \in L^p(\mathbf{R}^n) - W^{1,p}(A), \end{cases}$$

for which

$$|z|^p \leq f(x, s, z) \leq c\,[|z|^p + |s|^p + 1]$$

for every $x \in \mathbf{R}^n$, $s \in \mathbf{R}$, $z \in \mathbf{R}^n$.

We denote by $\mathcal{G}$ the class of all *decreasing local* functionals on $L^p(\mathbf{R}^n)$ which are *measures*.

Examples of functionals of the class $\mathcal{G}$ are the obstacle functionals

$$G(u, A) = \begin{cases} 0 & \text{if } \tilde{u} \geq \psi \quad \lambda\text{-a.e. on } A, \\ +\infty & \text{otherwise,} \end{cases}$$

where λ is any countably sub-additive set function, for instance the $(1, p)$-capacity.

By a *chain* of elements of $\mathcal{A}$ we mean a family $(A_t)_{t \in \mathbf{R}}$ of elements of $\mathcal{A}$ such that $A_s \subset\subset A_t$ for every $s, t \in \mathbf{R}$, with $s < t$.

We say that subset $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is *rich* if, for every chain $(A_t)_{t \in \mathbf{R}}$ of elements of $\mathcal{A}$, the set $\{t \in \mathbf{R} : A_t \notin \tilde{\mathcal{A}}\}$ is at most countable.

Let X be a topological space, let $\{F_h\}$ be a sequence of functions from X into $\overline{\mathbf{R}}$, let $x \in X$ and let $t \in \overline{\mathbf{R}}$. Following E. De Giorgi and T. Franzoni [11] we say that

$$t = \Gamma(X^-) \lim_{h\to\infty} F_h(x)$$

if and only if

$$t = \sup_{U \in \mathcal{T}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y) = \sup_{U \in \mathcal{T}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y),$$

where $\mathcal{T}(x)$ denotes the family of all neighbourhoods of x in X.

THEOREM 1. *Let $c \geq 1$ be a constant. Let $\{F_h\}$ be a sequence of functionals of the class $\mathcal{F} = \mathcal{F}(c)$ and let $\{G_h\}$ be a sequence of functionals of the class $\mathcal{G}$. Then there exists an increasing sequence of integers $\{h_k\}$, a functional F_∞ of the class $\mathcal{F}$, a functional G_∞ of the class $\mathcal{G}$, and a rich subset $\tilde{\mathcal{A}}$ of $\mathcal{A}$, such that*

$$F_\infty(u, A) = \Gamma(L^p(\mathbf{R}^n)^-) \lim_{k\to\infty} F_{h_k}(u, A),$$

$$F_\infty(u, A) + G_\infty(u, A) = \Gamma(L^p(\mathbf{R}^n)^-) \lim_{k\to\infty} [F_{h_k}(u, A) + G_{h_k}(u, A)]$$

for every $u \in L^p(\mathbf{R}^n)$ and for every $A \in \tilde{\mathcal{A}}$.

The fact that G_∞ is a measure can be proved using some results of G. Dal Maso and L. Modica [9]. The fact that G_∞ is decreasing can be proved as in H. Attouch and C. Picard [1].

Conditions under which F_∞ can be written as an integral are given in G. Buttazzo and G. Dal Maso [3]. The following integral representation theorem for the functional G_∞ is new. The proof will appear in G. Dal Maso [7].

THEOREM 2. *Let F_∞ and G_∞ be a the functionals given by Theorem 1. Assume that for every $A \in \mathcal{A}$ the function $u \to F_\infty(u, A)$ is strongly continuous on $W^{1,p}(\mathbf{R}^n)$. Then there exist two non-negative Borel measures μ and ν, and a non-negative Borel function $g : \mathbf{R}^n \times \mathbf{R} \to [0, +\infty]$, such that:*

(*a*) *for every $A \in \mathcal{A}$ and for every $u \in L^p(\mathbf{R}^n) \cap W^{1,p}_{loc}(A)$*

$$G_\infty(u, A) = \int_A g(x, \tilde{u}(x))\, d\mu(x) + \nu(A);$$

(*b*) *μ is a Radon measure, and $\mu \in W^{-1,q}(\mathbf{R}^n)$, with $p^{-1} + q^{-1} = 1$;*

(*c*) *for every $x \in \mathbf{R}^n$ the function $s \mapsto g(x, s)$ is decreasing and lower semi-continuous on $\mathbf{R}$.*

If the integrands f_h satisfy the conditions of G. Buttazzo and G. Dal Maso [3] (theorem 4.4 and remark 4.7) and the function

$$u \mapsto \int_A b(x, u(x))\, dx$$

is continuous on $L^p(\mathbf{R}^n)$, then the convergence of the minimum points and of the minimum values for the problems $(\mathscr{P}_h)$, considered in the introduction, follows from the general theory of Γ-convergence (see E. De Giorgi and T. Franzoni [12], section 2).

2. EXAMPLES

Throughout this section $n = p = 2$. For every $h \in \mathbf{N}$ we denote by E_h the union of all open balls in $\mathbf{R}^2$ of radius e^{-h^2} centred at the points of the form $(i/h, j/h)$, with $i, j \in \mathbf{Z}$.

In the following example the functionals G_h are obstacle functionals, the functionals F_h are equal to F_∞ and the functional G_∞ is not convex.

Example 1. Let $b : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $0 < \inf b < \sup b < +\infty$. We define

$$B(t) = \int_0^t b(s)\, ds .$$

For every $u \in L^2(\mathbf{R}^2)$ and for every $A \in \mathscr{A}$ we set

$$F(u, A) = \begin{cases} \int_A b(u)^2 |Du|^2 dx & \text{if } u \in L^2(\mathbf{R}^2) \cap W^{1,2}(A), \\ +\infty & \text{if } u \in L^2(\mathbf{R}^2) - W^{1,2}(A), \end{cases}$$

$$G_h(u, A) = \begin{cases} 0 & \text{if } \tilde{u} \geq 0 \quad \text{on } A \cap E_h, \\ +\infty & \text{otherwise,} \end{cases}$$

$$G_\infty(u, A) = 2\pi \int_A [B(u) \wedge 0]^2 dx .$$

There exists a rich subset $\tilde{\mathscr{A}}$ of $\mathscr{A}$, containing all bounded open sets with a lipschitzian boundary, such that

$$F(u, A) + G_\infty(u, A) = \Gamma(L^2(\mathbf{R}^2)^-) \lim_{h \to \infty} [F(u, A) + G_h(u, A)]$$

for every $u \in L^2(\mathbf{R}^2)$ and for every $A \in \tilde{\mathscr{A}}$.

It is enough to observe that

$$F(u, A) = \begin{cases} \int_A |D(B \circ u)|^2 \, dx & \text{if } u \in L^2(\mathbf{R}^2) \cap W^{1,2}(A), \\ +\infty & \text{if } u \in L^2(\mathbf{R}^2) - W^{1,2}(A), \end{cases}$$

$$G_h(u, A) = G_h(B \circ u, A).$$

Thus the result follows from L. Carbone and F. Colombini [4], proposition 3.1.

Note that, if $b(s) = 2 + \cos(s)$, then $B(t) = 2t + \sin(t)$, hence the functional G_∞ is not convex.

In the following example the functionals F_h are rapidly oscillating in u.

Example 2. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(0) = 0$ and, for $t \neq 0$, by

$$f(t) = \min \left\{ |t| \int_0^{1/|t|} [|v'|^2 + \sin^2(2\pi v)] \, ds : v \in C^1, v(0) = 0, v\left(\frac{1}{|t|}\right) = 1. \right.$$

For every $u \in L^2(\mathbf{R}^2)$ and for every $A \in \mathscr{A}$ set

$$F_h(u, A) = \begin{cases} \int_A [|Du|^2 + \sin^2(2\pi h u)] \, dx & \text{if } u \in L^2(\mathbf{R}^2) \cap W^{1,2}(A), \\ +\infty & \text{if } u \in L^2(\mathbf{R}^2) - W^{1,2}(A), \end{cases}$$

$$G_h(u, A) = \begin{cases} 0 & \text{if } \tilde{u} \geq 0 \quad \text{on } A \cap E_h, \\ +\infty & \text{otherwise,} \end{cases}$$

$$G_\infty(u, A) = 2\pi \int_A (u \wedge 0)^2 \, dx.$$

There exists a rich subset $\tilde{\mathscr{A}}$ of $\mathscr{A}$, containing all bounded open sets with a lipschitzian boundary, such that

$$F_\infty(u, A) = \Gamma(L^2(\mathbf{R}^2)^-) \lim_{h \to \infty} F_h(u, A)$$

$$F_\infty(u, A) + G_\infty(u, A) = \Gamma(L^2(\mathbf{R}^2)^-) \lim_{h \to \infty} [F_h(u, A) + G_h(u, A)]$$

for every $u \in L^2(\mathbf{R}^2)$ and for every $A \in \tilde{\mathscr{A}}$.

The first equality is proved in G. Buttazzo and G. Dal Maso [2]. The second one follows from Theorems 1 and 2 and from a comparison with the example of L. Carbone and F. Colombini (see [4], proposition 3.1).

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