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Calcolo delle variazioni. — *On the lower semicontinuity of certain integral functionals.* Nota di ENNIO DE GIORGI, GIUSEPPE BUTTAZZO e GIANNI DAL MASO (*), presentata (**) dal Corrisp. E. DE GIORGI.

RIASSUNTO. — Si dimostra che il funzionale $\int_\Omega f(u, \mathrm{D}u)\,\mathrm{d}x$ è semicontinuo inferiormente su $W^{1,1}_{loc}(\Omega)$, rispetto alla topologia indotta da $L^1_{loc}(\Omega)$, qualora l'integrando $f(s, p)$ sia una funzione non–negativa, misurabile in s, convessa in p, limitata nell'intorno dei punti del tipo $(s, 0)$, e tale che la funzione $s \mapsto f(s, 0)$ sia semicontinua inferiormente su $\mathbf{R}$.

INTRODUCTION

Let $n \geq 1$ be an integer and let Ω be an open subset of $\mathbf{R}^n$. For every $u \in W^{1,1}_{loc}(\Omega)$ we set $\mathrm{D}u = (\mathrm{D}_1 u, \cdots, \mathrm{D}_n u)$, where $\mathrm{D}_i u = \partial u/\partial x_i$. By "measurable" we always mean Lebesgue-measurable. For every $t \in \mathbf{R}$ we set $t^+ = \max\{t, 0\}$. For every function $f: \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ and for every $s \in \mathbf{R}$ we define

$$\alpha_f(s) = \limsup_{p\to 0} \frac{[f(s, 0) - f(s, p)]^+}{|p|}.$$

The aim of this paper is to prove the following theorem.

THEOREM 1. *Let* $f: \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ *be a function with the following properties*:

(a) *for every* $s \in \mathbf{R}$ *and* $p \in \mathbf{R}^n$ *we have* $f(s, p) \geq 0$;

(b) *for every* $p \in \mathbf{R}^n$ *the function* $s \mapsto f(s, p)$ *is measurable on* $\mathbf{R}$;

(c) *for every* $s \in \mathbf{R}$ *the function* $p \mapsto f(s, p)$ *is convex on* $\mathbf{R}^n$;

(d) *the function* $s \mapsto f(s, 0)$ *is lower semicontinuous on* $\mathbf{R}$;

(e) *the function* α_f *belongs to* $L^1_{loc}(\mathbf{R})$.

Then for every $u \in W^{1,1}_{loc}(\Omega)$ *the function* $x \mapsto f(u(x), \mathrm{D}u(x))$ *is measurable and the functional* $F: W^{1,1}_{loc}(\Omega) \mapsto [0, +\infty]$ *defined by*

$$F(u) = \int_\Omega f(u, \mathrm{D}u)\,\mathrm{d}x$$

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is lower semicontinuous on $W^{1,1}_{loc}(\Omega)$ *with respect to the topology induced by* $L^1_{loc}(\Omega)$.

REMARK 1. This theorem differs from other semicontinuity results (see [2], [5], [7] Chapter 4, [8] [9]) chiefly in that we do not assume that the function $s \mapsto f(s, p)$ is continuous or lower semicontinuous, except for $p = 0$. This allows us to include in a general framework the case of functionals of the form

$$\int_\Omega \left(\sum_{i,j=1}^n a_{i,j}(u) \, D_i u \, D_j u \right)^q dx$$

where $q \geq 1/2$ and $a_{i,j}$ are measurable functions such that

$$\sum_{i,j=1}^n a_{i,j}(s) \, p_i p_j \geq 0 \qquad \text{for every} \quad s \in \mathbf{R}, p \in \mathbf{R}^n.$$

REMARK 2. If f satisfies conditions (a), (b), (c) of Theorem 1, then condition (e) is satisfied whenever there exist $\varepsilon > 0$ and $\beta \in L^1_{loc}(\mathbf{R})$ such that $f(s, p) \leq \beta(s)$ for every $s \in \mathbf{R}$ and for every $p \in \mathbf{R}^n$ with $|p| \leq \varepsilon$.

REMARK 3. Hypothesis (e) in Theorem 1 cannot be dropped, as the following example shows. Let $n = 1$, $\Omega =]0, 1[$, and let f be defined by

$$f(s, p) = \begin{cases} \left[1 + \dfrac{p}{s}\right]^+ & \text{if } s \neq 0 \\ 1 & \text{if } s = 0. \end{cases}$$

For every $\varepsilon > 0$ let $u_\varepsilon(x) = \varepsilon - \varepsilon x$. Then (u_ε) converges to 0 as $\varepsilon \to 0$, but $F(u_\varepsilon) = 0$, whereas $F(0) = 1$. Note that f satisfies all conditions of Theorem 1 except (e).

PRELIMINARY LEMMAS.

For every $x, y \in \mathbf{R}^n$ we denote by $\langle x, y \rangle$ the scalar product of x and y and by $|x|$ the Euclidean norm of x.

LEMMA 1. *Let* $u \in W^{1,1}_{loc}(\Omega)$ *and let* E *be a Borel subset of* $\mathbf{R}$ *with meas* $(E) = 0$. *Then* $Du = 0$ *a.e. on* $u^{-1}(E)$.

Proof. The proof follows easily from a result of De La Vallée Poussin (see [3], [10]).

DEFINITION 1. We say that a function $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is an integrand if:

(*a*) for every $p \in \mathbf{R}^n$ the function $s \mapsto f(s, p)$ is measurable on $\mathbf{R}$;

(*b*) for every $s \in \mathbf{R}$ the function $p \mapsto f(s, p)$ is continuous on $\mathbf{R}$;

(*c*) the function $s \mapsto f(s, 0)$ is a Borel function.

DEFINITION 2. We say that two integrands f, g are equivalent integrands if there exists a Borel set $N \subseteq \mathbf{R}$ with meas(N) $= 0$ such that

(a) for every $s \in \mathbf{R} - N$ and $p \in \mathbf{R}^n$ we have $f(s, p) = g(s, p)$;

(b) for every $s \in \mathbf{R}$ we have $f(s, 0) = g(s, 0)$.

LEMMA 2. *If* f, g *are equivalent integrands and* $u \in W^{1,1}_{loc}(\Omega)$, *then* $f(u(x), Du(x)) = g(u(x), Du(x))$ *a.e. on* Ω.

Proof. It follows from Lemma 1.

LEMMA 3. *If* f *is an integrand and* $u \in W^{1,1}_{loc}(\Omega)$, *then the function* $x \mapsto f(u(x), Du(x))$ *is measurable on* Ω.

Proof. There exists a Borel function $g : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ such that f and g are equivalent integrands. The result now follows from Lemma 2.

LEMMA 4. *Let* $a : \mathbf{R} \to \mathbf{R}$ *be a Lipschitz continuous function and let* $b : \mathbf{R} \to \mathbf{R}$ *be a bounded measurable function such that* $a'(s) = b(s)$ *a.e. on* $\mathbf{R}$. *If* $u \in W^{1,1}_{loc}(\Omega)$ *and* $v = a \circ u$, *then* $v \in W^{1,1}_{loc}(\Omega)$ *and* $Dv = b(u)\, Du$ *a.e. on* Ω.

Proof. See [6] Lemma 1.2 and Lemma 1.5.

LEMMA 5. *Let* $b \in L^1(\mathbf{R}, \mathbf{R}^n)$ *and let* $a : \mathbf{R} \to \mathbf{R}^n$ *be defined by* $a(t) = \int_0^t b(s)\, ds$. *Let* $u \in W^{1,1}_{loc}(\Omega)$ *be a function such that*

$$\int_\Omega \langle b(u), Du \rangle^+ \, dx < +\infty .$$

Then, for every $\varphi \in C_0^\infty(\Omega)$ *with* $\varphi \geq 0$, *the function* $\langle b(u), Du \rangle \varphi$ *is in* $L^1(\Omega)$ *and*

$$\int_\Omega \langle b(u), Du \rangle \varphi \, dx = -\int_\Omega \langle a(u), D\varphi \rangle \, dx .$$

Proof. If b is bounded, the thesis follows from Lemma 4. In the general case it suffices to approximate b by the sequence (b_n) defined by

$$b_h(s) = \begin{cases} b(s) & \text{if } |b(s)| \leq h \\ 0 & \text{otherwise} . \end{cases}$$

LEMMA 6. *Let (f_n) be a sequence of non-negative measurable functions from $\mathbf{R}^n$ into $\mathbf{R}$ and let $f_\infty = \sup_h f_h$. Then for every open subset A of $\mathbf{R}^n$ we have*

$$\int_A f_\infty(x)\,dx =$$

$$= \sup_{k \in \mathbf{N}} \sup \left\{ \sum_{i=1}^{k} \int_{A_i} f_i(x)\,dx : A_1, \cdots, A_k \text{ pairwise disjoint open subsets of } A \right\}.$$

Proof. For every $k \in \mathbf{N}$ set $g_k = \sup\{f_i : i = 1, \cdots, k\}$; then by Beppo Levi's theorem we have

$$\int_A f_\infty(x)\,dx = \sup_{k \in \mathbf{N}} \int_A g_k(x)\,dx.$$

Now fix $k \in \mathbf{N}$; there exist measurable pairwise disjoint subsets $B_1, \cdots, B_k$ of A such that $g_k = f_i$ on B_i. Then

$$\int_A g_k(x)\,dx = \sum_{i=1}^{k} \int_{B_i} f_i(x)\,dx = \sup \left\{ \sum_{i=1}^{k} \int_{K_i} f_i(x)\,dx : K_i \subseteq B_i ;\ K_i \text{ compact} \right\} =$$

$$= \sup \left\{ \sum_{i=1}^{k} \int_{A_i} f_i(x)\,dx : A_1, \cdots, A_k \text{ pairwise disjoint open subsets of } A \right\}.$$

LEMMA 7. *Let (f_h) be a sequence of non-negative integrands and let $f_\infty = \sup_h f_h$. Set for every open subset A of Ω, every $u \in W^{1,1}_{loc}(A)$, and every $h \in \mathbf{N} \cup \{\infty\}$*

$$F_h(u, A) = \int_A f_h(u, Du)\,dx.$$

Suppose that for every $h \in \mathbf{N}$ and every open subset A of Ω the functional $F_h(\cdot, A)$ is $L^1_{loc}(A)$-lower semicontinuous. Then, for every open subset A of Ω the functional $F_\infty(\cdot, A)$ is $L^1_{loc}(A)$-lower semicontinuous.

Proof. It follows from Lemma 6.

PROOF OF THEOREM 1.

The proof of Theorem 1 is divided into two parts. In the first one we deal with the case $f(s, 0) = 0$ (considered in Lemma 10); then we shall use this partial result to prove the general case. The measurability of the function $x \mapsto f(u(x), Du(x))$ has already been proved in Lemma 3.

The functionals we are going to consider are defined in $W^1_{loc}(\Omega)$; when we say that a functional F is lower semicontinuous, we mean that F is lower semicontinuous on $W^1_{loc}(\Omega)$ with respect to the topology induced by $L^1_{loc}(\Omega)$. For every $B \subseteq \mathbf{R}$ we indicate by 1_B the characteristic function of B, defined by $1_B(s) = 1$ if $s \in B$ and $1_B(s) = 0$ if $s \in \mathbf{R} - B$.

LEMMA 8. *Let $b : \mathbf{R} \to \mathbf{R}^n$ be a measurable function and let $g : \mathbf{R} \to \mathbf{R}$ be a lower semicontinuous function with $g \leq 0$. Then the functional*

$$F(u) = \int_\Omega [g(u) + \langle b(u), Du\rangle]^+ \, dx$$

is lower semicontinuous.

Proof. First assume that b and g are bounded. For every $u \in W^{1,1}_{loc}(\Omega)$ we have

$$F(u) = \sup \left\{ \int_\Omega [g(u) + \langle b(u), Du\rangle] \, \varphi \, dx : \varphi \in C^\infty_0(\Omega) , 0 \leq \varphi \leq 1 \right\};$$

therefore it is enough to prove that for every $\varphi \in C^\infty_0(\Omega)$, with $\varphi \geq 0$, the functionals

$$G(u) = \int_\Omega g(u) \, \varphi \, dx$$

$$H(u) = \int_\Omega \langle b(u), Du\rangle \, \varphi \, dx$$

are lower semicontinuous. For G it is enough to apply Fatou's lemma. From Lemma 4 we obtain

$$H(u) = \int_\Omega \operatorname{div}(a \circ u) \, \varphi \, dx = - \int_\Omega \langle a(u), D\varphi\rangle \, dx$$

where $a(t) = \int_0^t b(s) \, ds$.

This implies that H is continuous on $W^{1,1}_{loc}(\Omega)$ with respect to the topology induced by $L^1_{loc}(\Omega)$.

If b or g are unbounded, let (b_h) be the sequence of functions defined by

$$b_h(s) = \begin{cases} b(s) & \text{if } |b(s)| \leq h \\ 0 & \text{otherwise} \end{cases}$$

and let (σ_h) be an increasing sequence of functions in $C_0^\infty(\mathbf{R})$ with $\sigma_h \geq 0$ and $\lim_h \sigma_h(s) = 1$ for every $s \in \mathbf{R}$. Since g is lower semicontinuous and $g \leq 0$, every function $\sigma_h(s)\, g(s)$ is bounded. By Beppo Levi's theorem

$$F(u) = \sup_{h \in \mathbf{N}} \int_\Omega [\sigma_h(u)\, g(u) + \langle \sigma_h(u)\, b_h(u), Du \rangle]^+ \, dx.$$

Therefore the lower semicontinuity of F follows from the result obtained in the bounded case.

LEMMA 9. *Let* $b : \mathbf{R} \to \mathbf{R}^n$ *be a measurable function and let* $g : \mathbf{R} \to \mathbf{R}$ *be a measurable function with* $g \leq 0$. *Then the functional*

$$F(u) = \int_\Omega [g(u) + \langle b(u), Du \rangle]^+ \, dx$$

is lower semicontinuous.

Proof. By Lusin's theorem there exists an increasing sequence (K_h) of compact subsets of $\mathbf{R}$ and a sequence (g_h) of continuous functions with $g_h \leq 0$, such that $g_h(s) = g(s)$ for every $s \in K_h$ and $\text{meas}(\mathbf{R} - E) = 0$, where $E = \bigcup_h K_h$. Since $g \leq 0$, using Lemma 2 and Beppo Levi's Theorem, we get

$$F(u) = \int_\Omega 1_E(u)\, [g(u) + \langle b(u), Du \rangle]^+ \, dx =$$

$$= \sup_{h \in \mathbf{N}} \int_\Omega [1_{K_h}(u)\, g_h(u) + \langle 1_{K_h}(u)\, b(u), Du \rangle]^+ \, dx$$

for every $u \in W^{1,1}_{\text{loc}}(\Omega)$. Since $g_h \leq 0$, the functions $1_{K_h}(s)\, g_h(s)$ are lower semicontinuous, thus the lower semicontinuity of F follows from Lemma 8.

LEMMA 10. *Assume that* f *satisfies conditions* (a), (b) (c) *of Theorem 1, and that* $f(s, 0) = 0$ *for every* $s \in \mathbf{R}$. *Then the functional*

$$F(u) = \int_\Omega f(u, Du)\, dx$$

is lower semicontinuous.

Proof. For every $s \in \mathbf{R}$ set

$$K(s) = \{(a, b) \in \mathbf{R} \times \mathbf{R}^n : f(s, p) \geq a + \langle b, p \rangle \ \forall p \in \mathbf{R}^n\}.$$

By the measurable selection theorem (see [1] Th. III, 30 page 80) there exist a sequence (a_h) of measurable functions from $\mathbf{R}$ into $\mathbf{R}$, and a sequence (b_h)

of measurable functions from $\mathbf{R}$ into $\mathbf{R}^n$, such that for every $s \in \mathbf{R}$ the set $\{(a_h(s), b_h(s)) : h \in \mathbf{N}\}$ is dense in $K(s)$. Then for every $s \in \mathbf{R}, p \in \mathbf{R}^n$

$$(1) \qquad f(s, p) = \sup\{[a + \langle b, p\rangle]^+ : (a, b) \in K(s)\} = \sup_{h \in \mathbf{N}} [a_h(s) + \langle b_h(s), p\rangle]^+$$

Since $f(s, 0) = 0$, by (1) we have $a_h(s) \leq 0$. Thus the lower semicontinuity of F follows from Lemma 9 and from Lemma 7.

LEMMA 11. *Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Under the assumptions of Lemma 10 the functional*

$$F(u) = \int_\Omega f(u, Du)\, \varphi\, dx$$

is lower semicontinuous.

Proof. For every $h, k \in \mathbf{N}$ let $\Omega_{h,k} = \{x \in \Omega : \varphi(x) > k\, 2^{-h}\}$ and let

$$\varphi_h(x) = 2^{-h} \sum_{k=1}^{4^h} 1_{\Omega_{h,k}}(x).$$

The sequence (φ_h) is increasing and $\varphi = \sup_{h \in \mathbf{N}} \varphi_h$. Then

$$F(u) = \sup_{h \in \mathbf{N}} \int_\Omega f(u, Du)\, \varphi_h\, dx = \sup_{h \in \mathbf{N}} 2^{-h} \sum_{k=1}^{4^h} \int_{\Omega_{h,k}} f(u, Du)\, dx.$$

Thus the lower semicontinuity of F follows from Lemma 10.

Proof of Theorem 1. Assume first that $\alpha_f \in L^1(\mathbf{R})$. For every $s \in \mathbf{R}$ let $\partial f(s, 0)$ be the subdifferential at the point $p = 0$ of the convex function $p \mapsto f(s, p)$ and let $b(s)$ be the element of $\partial f(s, 0)$ such that

$$|b(s)| = \min\{|q| : q \in \partial f(s, 0)\}.$$

It is known that $b : \mathbf{R} \to \mathbf{R}^n$ is measurable (see [4], Th. 1.2, page 236) and that $|b(s)| = \alpha_f(s)$ for every $s \in \mathbf{R}$. Since

$$(2) \qquad f(s, p) \geq f(s, 0) + \langle b(s), p\rangle$$

for every $s \in \mathbf{R}, p \in \mathbf{R}^n$, the function

$$(3) \qquad g(s, p) = f(s, p) - f(s, 0) - \langle b(s), p\rangle$$

satisfies all conditions of Lemma 10.

Let (u_h) be a sequence in $W^{1,1}_{loc}(\Omega)$ converging in $L^1_{loc}(\Omega)$ to a function $u_\infty \in W^{1,1}_{loc}(\Omega)$; we have to prove that

$$(4) \qquad F(u_\infty) \leq \liminf_h F(u_h).$$

If the right-hand side is $+\infty$ the inequality is trivial. So we may assume that $\liminf_h F(u_h) < +\infty$ and that $F(u_h) < +\infty$ for every $h \in \mathbf{N}$. Since $f(s, p) \geq 0$ by (2) we obtain

$$\int_\Omega \langle b(u_h), Du\rangle^+ dx \leq F(u_h) < +\infty .$$

Since the function $\langle b(s), p\rangle^+$ satisfies all conditions of Lemma 10 we have

$$\int_\Omega \langle b(u_\infty), Du_\infty\rangle^+ dx \leq \liminf_h \int_\Omega \langle b(u_h), Du_h\rangle^+ dx \leq \liminf_h F(u_h) < +\infty.$$

Let $\varphi \in C_0^\infty(\Omega)$ with $0 \leq \varphi \leq 1$. For every $s \in \mathbf{R}$ set

$$a(t) = \int_0^t b(s)\, ds ;$$

by Lemma 5

$$\int_\Omega \langle b(u_h), Du_h\rangle \varphi\, dx = -\int_\Omega \langle a(u_h), D\varphi\rangle dx \tag{5}$$

for every $h \in \mathbf{N} \cup \{\infty\}$. By Lemma 11

$$\int_\Omega g(u_\infty, Du_\infty) \varphi\, dx \leq \liminf_h \int_\Omega g(u_h, Du_h) \varphi\, dx . \tag{6}$$

Since the function $s \mapsto f(s, 0)$ is lower semicontinuous, by Fatou's Lemma

$$\int_\Omega f(u_\infty, 0) \varphi\, dx \leq \liminf_h \int_\Omega f(u_h, 0) \varphi\, dx . \tag{7}$$

Since a is continuous and bounded, from (5) we get

$$\int_\Omega \langle b(u_\infty), Du_\infty\rangle \varphi\, dx = \lim_h \int_\Omega \langle b(u_h), Du_h\rangle \varphi\, dx . \tag{8}$$

From (3), (6), (7), (8) we obtain

$$\int_\Omega f(u_\infty, Du_\infty) \varphi\, dx \leq \liminf_h \int_\Omega f(u_h, Du_h) \varphi\, dx \leq \liminf_h F(u_h) .$$

Since

$$F(u_\infty) = \sup \left\{ \int_\Omega f(u_\infty, Du_\infty)\, \varphi \, dx : \varphi \in C_0^\infty(\Omega), 0 \leq \varphi \leq 1 \right\}$$

we get (4) and the Theorem is proved in the case $\alpha_f \in L^1(\mathbf{R})$.

In the general case $\alpha_f \in L^1_{loc}(\mathbf{R})$, let (σ_h) be an increasing sequence of functions of $C_0^\infty(\mathbf{R})$ with $\sigma_h \geq 0$ and $\lim_h \sigma_h(s) = 1$ for every $s \in \mathbf{R}$, let $f_h(s, p) = \sigma_h(s) f(s, p)$ for every $s \in \mathbf{R}, p \in \mathbf{R}^n$, and let $F_h(u) = \int_\Omega f_h(u, Du)\, dx$. For every $u \in W^{1,1}_{loc}(\Omega)$ we have

$$F(u) = \sup_h F_h(u).$$

Since $\alpha_{f_h} \in L^1(\mathbf{R})$ the functionals F_h are lower semicontinuous; hence F is lower semicontinuous and the Theorem is proved.

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