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Teoria dei controlli. — *On the well-posedness of some optimal control problems.* Nota [*] di AUGUSTO VISINTIN [**], presentata dal Corrisp. E. MAGENES.

RIASSUNTO. — Si considerano problemi di controllo ottimale con una dipendenza non lineare tra il controllo e lo stato. Si mostra come in certi casi la continuità di tale dipendenza, quindi la buona posizione nel senso di Tychonov, è connessa alla forma del funzionale costo. In particolare si esamina un problema di Stefan a due fasi con controllo distribuito nel termine di sorgente.

INTRODUCTION

Consider the general formulation of a problem of optimal control (see [2] e.g.). We have a space U, a subset $\tilde{U}$ (set of admissible controls), another space Y (space of possible states of the system), an application $\Phi : \tilde{U} \rightarrow Y$ (the state equation $y = \Phi(v)$ may be given implicitly by a boundary value problem for a partial differential equation, e.g.) with range $\tilde{Y} := \Phi(\tilde{U})$ (set of admissible states), and finally a functional $\Psi : \tilde{U} \times \tilde{Y} \rightarrow \mathbf{R}$.

The problem is to minimize the cost functional $J(v) := \Psi(v, \Phi(v))$ over $\tilde{U}$, or equivalently to minimize $\Psi(v, y)$ over $G := \{(v, y) \in \tilde{U} \times \tilde{Y} \mid y = \Phi(v)\}$ (= graph of Φ).

In a more general framework (see [3]) Ψ is defined only on $\tilde{U} \times (\tilde{Y} \cap \tilde{Y}')$, where Y' is a subset of Y; in this case Ψ is minimized over $G' := G \cap (U \times Y')$.

So much for the set structure. For the purposes of analysis, Φ is usually assumed to be continuous, the topologies of U and Y being related to the form of the state equation. However, in some cases the minimization problem has a solution even if Φ is not continuous w.r.t. any natural topology.

Here we show some examples in which the form of J determines the topology of U for which the minimization problem is well-posed in the generalized sense of Tychonov; of course this can be useful for the numerical approximation.

The underlying scheme is the following. Assume that $\tilde{Y}$ is compact w.r.t. a natural topology $\mathscr{T}$; then for any sequence $\{v_n\}$ in $\tilde{U}$ there exists a $y \in \tilde{Y}$ and a subsequence $\{v_n\}$ such that

$$(1) \qquad y_{n'} = \Phi(v_{n'}) \rightarrow y = \Phi(v) \quad \text{in } \mathscr{T};$$

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in general $v_{n'}$ does not converge to v in any non-trivial topology, not even for a subsequence. Now assume that Ψ does not depend explicitly on v and that it is lower semi-continuous of $\tilde{Y}$; if $\{v_n\}$ is a minimizing sequence for J, then one easily gets

$$J(v_{n'}) = \Psi(y_{n'}) \to J(v) = \Psi(y) . \tag{2}$$

In certain cases (2) and the form of Φ, i.e. of the state equation, allow us to prove that $v_{n'}$ converges to v in a non-trivial topology; for instance, and this is the technique used in the examples below, (2) may be used to improve the convergence in (1). As we shall show, different types of convergence of $v_{n'}$ to v may correspond to different choices of J.

In the examples given here, the state equation consists of a free boundary problem. In the developments we use the well-known property that weak convergence and convergence of the norms entail strong convergence in L^p-spaces, for $1 < p < \infty$. Generalizations can be constructed using functionals $J : B \to \mathbf{R}$ (with B Banach space) such that if $v_n \to v$ weakly in B and $J(v_n) \to J(v)$, then $v_n \to v$ strongly in B. Sufficient conditions for this last property are given in Theorem 3 of [5].

1. STEFAN PROBLEM WITH SOURCE CONTROL

Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$, $T > 0$; set $Q := \Omega \times]0, T[$, $U := L^\infty(Q)$, $\tilde{U} := \{v \in L^\infty(Q) \mid c_1 \leqq v \leqq c_2 \text{ a.e. in } Q\}$, with c_1, c_2 constants and $0 < c_1 < c_2$. Let V be a subspace of $H^1(\Omega)$ and $A: V \to V'$ be defined by ${}_{V'}\langle Au, u\rangle_V = \int_\Omega \nabla u \cdot \nabla v dx$, $\forall u, v \in V$. Set also $H(\xi) = \{0\}$ if $\xi > 0$, $H(0) = [0, 1]$, $H(\xi) = \{1\}$ if $\xi > 0$.

Finally let

$$f \in L^2(0, T : V') \quad , \quad g \in L^2(Q) \quad , \quad \alpha, \beta \in L^2(Q) \quad , \quad z^0 \in V' . \tag{1.1}$$

For any $v \in \tilde{U}$, consider the following problem

(P) Find $w \in L^2(0, T; V)$, $\chi \in L^\infty(Q)$ such that

$$\chi \in H(w) \qquad \text{a.e. in } Q \tag{1.2}$$

$$\frac{\partial}{\partial t}(w + \chi) + Aw = f + g\chi + v(\alpha\chi + \beta) \quad \text{in } V' \text{ , a.e. in }]0, T[\tag{1.3}$$

$$(w + \chi)_{|t=0} = z^0 \qquad \text{in } \Omega . \tag{1.4}$$

Remark. (1.3) yields $w + \chi \in H^1(0, T; V')$; this gives a meaning to (1.4).

(P) is the weak formulation of a two-phase Stefan problem; w represents temperature and $w + \chi$ enthalphy, the latent heat and other physical constants being assumed equal to 1. For more information about this type of problem we refer to [1] e.g.

The above situation corresponds to a fairly general class of distributed controls for the source of the Stefan problem. We notice that if the control should not appear linearly and if v should be replaced by a term of the form $g(v)$ with g continuous, then we could easily reduce to the above situation replacing v by $\tilde{v} := g(v)$ and changing $\tilde{\mathrm{U}}$ accordingly.

The following result is well-known:

LEMMA 1. $\forall\, v \in \tilde{\mathrm{U}}$, *under assumption* (1.1) *problem* (P) *has one and only one solution.*

If moreover $\left(\textit{setting}\ \lambda^+ := \dfrac{|\lambda| + \lambda}{2}\ ,\quad \lambda^- := \dfrac{|\lambda| - \lambda}{2}\ ,\quad \forall \lambda \in \mathbf{R}\right)$

$$\left\{\begin{array}{l} f \in \mathrm{L}^2(\mathrm{Q}) \\ (z_0 - 1)^+ - z_0^- \in \mathrm{V}\,, \end{array}\right. \tag{1.5}$$

then the solution of (P) *has the further regularity*

$$w \in \mathrm{H}^1(0\,,\mathrm{T}\,;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0\,,\mathrm{T}\,,\mathrm{V})\,. \quad \blacksquare \tag{1.6}$$

In (P) the control is distributed and acts in the source; moreover it is phase-dependent. From now on, we shall always assume (1.1) and

$$\alpha\,,\ \alpha + \beta \in [\gamma_1\,,\gamma_2] \text{ a.e. in } \Omega\ ,\quad \text{with } \gamma_1\,,\gamma_2\text{: constant, } 0 < \gamma_1 < \gamma_2 \tag{1.7}$$

(this last condition could be easily weakened).

$$\forall v \in \tilde{\mathrm{U}}\,,\quad \text{set } \Phi(v) := (w\,,\chi)\text{: solution of (P)}\,.$$

LEMMA 2. $\tilde{\mathrm{Y}} := \Phi(\mathrm{U})$ *is compact w.r.t. the weak star topology of* $\mathrm{L}^2(0\,,\mathrm{T};\mathrm{V}) \times \mathrm{L}^\infty(\mathrm{Q})$.

Proof. Let $\{(w_n\,,\chi_n) = \Phi(v_n)\}$ be a generic sequence in $\tilde{\mathrm{Y}}$, that is

$$\chi_n \in \mathrm{H}(w_n) \qquad \text{a.e. in Q} \tag{1.8}$$

$$\frac{\partial}{\partial t}(w_n + \chi_n) + \mathrm{A}w_n = f + g\chi_n + v_n(\alpha\chi_n + \beta) \text{ in V}'\,, \text{ a.e. in }]0\,,\mathrm{T}[\tag{1.9}$$

$$(w_n + \chi_n)|_{t=0} = z^0 \qquad \text{in } \Omega\,. \tag{1.10}$$

Multiply (1.9) against w_n and integrate w.r.t. time; by a standard procedure we get

(1.11) $$\| w_n \|_{L^\infty(0,T\,;\,L^2(\Omega)) \cap L^2(0,T,V)} \leqq \text{Const.}$$ (i.e. constant independent of n),

then comparing in (1.9) we have

(1.12) $$\| w_n + \chi_n \|_{H^1(0,T\,;\,V')} \leqq \text{Const.};$$

moreover of course

(1.13) $$\| \chi_n \|_{L^\infty(Q)}\,, \quad \| v_n (\alpha\chi_n + \beta) \|_{L^\infty(Q)} \leqq \text{Const.}$$

Therefore, there exist w, χ, ξ such that, possibly taking subsequences,

(1.14) $w_n \to w$ weakly star in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; V)$

(1.15) $w_n + \chi_n \to w + \chi$ weakly star in $L^\infty(0, T; L^2(\Omega))$, weakly in $H^1(0, T; V')$

(1.16) $\chi_n \to \chi$ weakly star in $L^\infty(Q)$

(1.17) $v_n(\alpha\chi_n + \beta) \to \xi$ weakly star in $L^\infty(Q)$.

We have $c_1(\alpha\chi_n + \beta_n) \leqq v_n(\alpha\chi_n + \beta) \leqq c_2(\alpha\chi_n + \beta)$, whence $c_1(\alpha\chi + \beta) \leqq \xi \leqq c_2(\alpha\chi + \beta)$ a.e. in Q; therefore setting $v := \dfrac{\xi}{\alpha\chi + \beta}$ a.e. in Q we have $v \in \tilde{U}$ and

(1.18) $$\xi = v(\alpha\chi + \beta) \qquad \text{a.e. in } Q\,.$$

By a standard procedure based on the maximal monotonicity of H, (1.8), (1.14) and (1.15) yield (1.2). Taking $n \to \infty$ in (1.9), (1.10) we get (1.3), (1.4). Therefore $(w, \chi) = \Phi(v)$. ∎

Remark that $\{v_n\}$ does not necessarily converge to v, not even for a subsequence.

1st *example*. $\forall v \in \tilde{U}$, let $(w, \chi) = \Phi(v)$ and set $J_1(v) := \| \chi - \chi_d \|_{L^p(Q)}$, with $1 < p < \infty$ and χ_d given in $L^p(Q)$ (not necessarily $0 \leqq \chi_d \leqq 1$).

The minimization of J_1 corresponds to controlling the evolution of the free boundary of the Stefan problem (P). Different problems of control of the free boundary for one or two-phase Stefan problems by means of boundary controls have been studied by Saguez (see [4]).

PROPOSITION 1. *Any sequence in* $\tilde{U}$ *minimizing* J_1 *has a subsequence which converges to a minimum of* J_1 *w.r.t. the weak star topology of* $L^\infty(Q)$ *(Generalized well-posedness in the sense of Tychonov).*

Proof. Let $\{v_{1n} \in \tilde{U}\}_{n \in N}$ be a minimizing sequence for J_1; by Lemma 2 there exists at least one $v_1 \in \tilde{U}$ such that, setting $(\chi_{1n}, w_{1n}) = \Phi(v_{1n})\ \forall n$, $(\chi_1, w_1) = \Phi(v_1)$ and possibly taking subsequences

$$\chi_{1n} \to \chi_1 \qquad \text{weakly star in } L^\infty(Q) \tag{1.19}$$

$$w_{1n} \to w_1 \qquad \text{weakly in } L^2(0, T; V) \tag{1.20}$$

whence

$$v_{1n}(\alpha\chi_{1n} + \beta) = \frac{\partial}{\partial t}(w_{1n} + \chi_{1n}) + Aw_{1n} - f - g\chi_n \to \frac{\partial}{\partial t}(w_1 + \chi_1) + \\ + Aw_1 - f - g\chi = v_1(\alpha\chi_1 + \beta) \qquad \text{weakly star in } L^\infty(Q). \tag{1.21}$$

By the lower semi-continuity of norms,

$$\inf_{\tilde{U}} J_1 = \lim_{n \to 0} \| \chi_{1n} - \chi_d \|_{L^p(Q)} \geqq \| \chi_1 - \chi_d \|_{L^p(Q)} \geqq \inf_{\tilde{U}} J_1, \tag{1.22}$$

hence

$$\| \chi_{1n} - \chi_d \|_{L^p(Q)} \to \| \chi_1 - \chi_d \|_{L^p(Q)} \tag{1.23}$$

and then by (1.19), applying a well-known property of L^p-spaces,

$$\chi_{1n} \to \chi_1 \qquad \text{strongly in } L^p(Q) \text{ and a.e. in } Q \text{ for a subsequence;} \tag{1.24}$$

(1.7) yields $\dfrac{1}{\gamma_2} \leqq \dfrac{1}{\alpha\chi_{1n} + \beta} \leqq \dfrac{1}{\gamma_1}$, therefore

$$\frac{1}{\alpha\chi_{1n} + \beta} \to \frac{1}{\alpha\chi_1 + \beta} \qquad \text{weakly star in } L^\infty(Q) \text{ and a.e. for a sub-} \tag{1.25}$$

sequence, hence strongly in $L^q(Q)$, $\forall q < +\infty$.

Finally by (1.21) and (1.25)

$$v_{1n} = v_{1n}(\alpha\chi_{1n} + \beta) \cdot \frac{1}{\alpha\chi_{1n} + \beta} \to v_1(\alpha\chi_1 + \beta) \cdot \frac{1}{\alpha\chi_1 + \beta} = v_1 \tag{1.26}$$

weakly star in $L^\infty(Q)$. ∎

2^{nd} *example.* $\forall v \in \tilde{U}$ let $(w, \chi) = \Phi(v)$ and set $J_2(v) := \| v(\alpha\chi + \beta) - \xi_d \|_{L^p(Q)}$ with $1 < p < \infty$ and ξ_d given in $L^p(\Omega)$.

The minimization of J_2 corresponds to controlling the phase-dependent source of (1.3) (1).

We introduce a new convergence in $\tilde{U}$:

$$(1.27) \qquad v_n \to v \text{ in } \tau \text{ if and only if } \frac{1}{v_n} \to \frac{1}{v} \text{ weakly star in } L^\infty(Q).$$

PROPOSITION 2. *Any sequence in* $\tilde{U}$ *minimizing* J_2 *has a subsequence which* τ*-converges to a minimum of* J_2 *(Generalized well-posedness in the sense of Tychonov).*

Proof. Let $\{v_{2n} \in \tilde{U}\}$ be a minimizing sequence for J_2; still by Lemma 2 there exists at least one $v_2 \in \tilde{U}$ such that, setting $(\chi_{2n}, w_{2n}) = \Phi(v_{2n})\ \forall n$, $(\chi_1, w_1) = \Phi(v_1)$ and possibly taking a subsequence,

$$(1.28) \qquad \chi_{2n} \to \chi_2 \qquad \text{weakly star in } L^\infty(Q)$$

$$(1.29) \qquad w_{2n} \to w_2 \qquad \text{weakly in } L^2(0, T; V),$$

whence as for (1.21)

$$(1.30) \qquad v_{2n}(\alpha\chi_{2n} + \beta) \to v_2(\alpha\chi_2 + \beta) \quad \text{weakly star in } L^\infty(Q).$$

We have

$$(1.31) \qquad \inf_{\tilde{U}} J_2 = \lim_{n\to\infty} \| v_{2n}(\alpha\chi_{2n} + \beta) - \xi_d \|_{L^p(Q)} \geqq \| v_2(\alpha\chi_2 + \beta) - \xi_d \|_{L^p(Q)} \geqq$$
$$\geqq \inf_{\tilde{U}} J_2$$

hence

$$(1.32) \qquad \| v_{2n}(\alpha\chi_{2n} + \beta) - \xi_d \|_{L^p(Q)} \to \| v_2(\alpha\chi_2 + \beta) - \xi_d \|_{L^p(Q)}$$

and then by (1.30), applying the same property of L^p-spaces used before,

$$(1.33) \qquad v_{2n}(\alpha\chi_{2n} + \beta) \to v_2(\alpha\chi_2 + \beta) \text{ strongly in } L^p(Q) \text{ and a.e. in } Q \text{ for a subsequence;}$$

by definition of $\tilde{U}$, $v_{2n}(\alpha\chi_{2n} + \beta) \geqq c_1 > 0$, hence

$$(1.34) \qquad \frac{1}{v_{2n}(\alpha\chi_{2n} + \beta)} \to \frac{1}{v_2(\alpha\chi_2 + \beta)} \text{ weakly star in } L^\infty(Q) \text{ and a.e. in } Q \text{ for a subsequence;}$$

(1) In order to fit this setting into the scheme indicated in the introduction, one should replace Φ by $\tilde{\Phi}(v) := (w, v(\chi + 1))$.

by (1.28) and (1.34) we have

$$(1.35)\quad \frac{1}{v_{2n}} = \frac{1}{v_{2n}(\alpha\chi_{2n}+\beta)}\cdot(\alpha\chi_{2n}+\beta) \to \frac{1}{v_2(\alpha\chi_2+\beta)}\cdot(\alpha\chi_2+\beta) = \frac{1}{v_2}$$

weakly star in $L^\infty(Q)$.

that is $v_{2n} \to v_2$ in τ. ∎

(1.35) is to be compared with (1.26).

Remark. Dealing with Example 1 one is tempted to replace J_1 by $J_1^*(v) := J_1(v) + \|v\|_{L^p(Q)}$, and similarly J_2 by $J_2^*(v) := J_2(v) + \left\| \frac{1}{v} \right\|_{L^r(Q)}$ for Example 2 (with $1 < p < \infty$). However J_1^* and J_2^* are not lower semi-continuous, as Φ is not continuous a priori, not even at the minimizing argument; therefore in this case the previous procedure cannot be used for proving the existence of a minimum.

2. Other examples

We consider the control of an obstacle problem. Let Ω, V, A, H be as in section 1; set $U := L^\infty(\Omega)$, $\tilde{U} := \{v \in L^\infty(\Omega) \mid c_1 \leqq v \leqq c_2 \text{ a.e. in } \Omega\}$, with c_1, c_2 constants and $0 < c_1 < c_2$.

$\forall v \in \tilde{U}$, let $\Phi(v) = (w, \chi) \in Y := V \times L^\infty(\Omega)$ be the solution of the system

$$(2.1)\qquad \chi \in H(w) \qquad \text{a.e. in } \Omega$$

$$(2.2)\qquad v(\chi+1) + Aw = f + g\chi \quad \text{in } V',$$

with $f \in V'$, $g \in L^2(\Omega)$ given (more generally one can replace $\chi + 1$ by $\alpha\chi + \beta$, as in § 1). This is equivalent to a variational inequality and has one and only one solution. By a procedure similar to that used for Lemma 2, one can prove that $\tilde{Y} := \Phi(\tilde{U})$ is compact w.r.t. the weak star topology of $V \times L^\infty(\Omega)$.

Examples similar to those of section 1 can be introduced.

We consider a last class of optimization problems.

Let $\Omega, Q, \tilde{U}, V, A$ be defined as in section 1; let $f \in L^2(0, T; V')$. $\forall v \in \tilde{U}$, let $w = \Phi(v) \in Y := L^2(0, T; V)$ be the solution of the problem

$$(2.3)\qquad v(w+1) + Aw = f \qquad \text{in } V', \quad \text{a.e. in }]0, T[.$$

This is a family of elliptic equations, parametrized by $t \in]0, T[$. Assume that $f \geq 0$ in the sense of $\mathcal{D}'(Q)$; then $w \geq 0$ a.e. in Q by the maximum principle, hence if $v'(w+1) = v''(w+1)$ a.e. in Q then $v' = v''$ a.e. in Q, that is Φ is injective. $\tilde{Y} := \Phi(\tilde{U})$ is bounded in Y, as it is easy to check multiplying (2.3) against w; $\tilde{Y}$ is also closed w.r.t. the weak topology of Y, as can be proved by a procedure similar to that used in the proof of lemma 2. Thus $\tilde{Y}$ is weakly compact in $L^2(0, T; V)$.

One can consider two examples similar to the previous ones:

(i) $J_1(v) = \| \Phi(v) - w_d \|_V$, with $w_d \in V$ given. The corresponding minimization problem is well-posed in the generalized sense of Tychonov w.r.t. the weak star topology of $L^\infty(Q)$.

(ii) $J_2(v) := \| v[\Phi(v) + 1] - \xi_d \|_{L^2(\Omega)}$ with $\xi_d \in L^2(\Omega)$ given. The corresponding minimization problem is well-posed in the generalized sense of Tychonov w.r.t. the topology induced by the τ-convergence defined in (1.27).

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