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Uniqueness theorems for steady, compressible, heat-conducting fluids: exterior domains

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Fisica matematica. — *Uniqueness theorems for steady, compressible, heat-conducting fluids: exterior domains* (*). Nota II di MARIA-ROSARIA PADULA (**), presentata (***) dal Socio D. GRAFFI.

RIASSUNTO. — Si fornisce un teorema di unicità per moti stazionari regolari di fluidi compressibili, viscosi, termicamente conduttori, svolgentisi in regioni esterne a domini compatti della spazio fisico.

§ 1. This note continues the problem stated in note I concerning uniqueness of steady, compressible, heat-conducting, ideal polytropic fluid flows. Precisely, whereas in note I we considered motions occurring in a bounded region, here we prove a uniqueness theorem for regular motions occurring in a domain Ω exterior to a compact region Ω_0 of the physical three dimensional space $\mathbf{R}^3$ (the case $\Omega = \mathbf{R}^3$ is allowed). The boundary $\partial\Omega$ is assumed rigid and of infinite thermal conductivity (velocity and temperature ascribed). The smoothness assumptions on solutions are the usual ones [1, 2] and include suitable differentiability and summability hypotheses on the solutions together with the existence of a strict positive lower bound for the density on each compact sub-region of Ω and for the temperature on the whole of Ω. Moreover, we shall make the following assumptions on the density ρ: either i) $\rho(x) = 0(|x|^{-2})$; or ii) $\rho \in L^3(\Omega)$. It is worth remarking explicitly that assumptions i) or ii) on ρ allow the finiteness of the total mass of the fluid, unlike the case of the uniqueness for *unsteady* motions where, up to data, the total mass *must* be infinite just as a consequence of the behaviour assumed at infinity on ρ [3, 4]. Finally, as observed also in note I the hypotheses on the fluid to be ideal and polytropic are by no means restrictive.

The plan of the work is the following. In section 2 we give some preliminary lemmas and define the regularity classes $\mathscr{I}_1, \mathscr{I}_2$ where uniqueness is proved. In section 3, we prove the uniqueness theorem which is stated in terms of suitable nondimensional parameters.

We end by noticing that, though the classes $\mathscr{I}_1$ and $\mathscr{I}_2$ are non empty, as shown in section 2, it would be desirable to provide existence in it. However, unlike the case of a bounded domain, this question appears more difficult in this case because one has to prove suitable decay to zero at infinity for the solutions.

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The uniqueness problem has already been stated in note I and therefore it will be omitted here. Furthermore, if no explicit mention is made, we shall use the same notation.

§ 2. We begin this section by introducing some notation. We set

$$g(r) = \begin{cases} 3 & |x| \equiv r \leq 1 \\ r^{-2} & |x| \equiv r > 1 . \end{cases}$$

Moreover, H denotes the Hilbert space of functions $w : \Omega \to \mathbf{R}$ vanishing on the boundary such that

$$\| w \|^2 = \int_\Omega g(r)\, w^2 \,\mathrm{d}x + \int_\Omega (\nabla w)^2 \,\mathrm{d}x .$$

As can easily be verified [5] the space H coincides with the completion of the set of indefinitely differentiable functions in Ω with support compact ($C_0^\infty(\Omega)$) with respect to the norm $\int_\Omega (\nabla w)^2 \,\mathrm{d}x$. Concerning the space H we have the following embeddings.

LEMMA 1. $H \subseteq L^6(\Omega)$, *i.e. there exists a constant* c_0 *such that*

$$| w |_6 \leq c_0 \, | \nabla w |_2$$

where $| \cdot |_p$ *denotes the usual* L^p*-norm.*

LEMMA 2. *For any* $w \in H$ *it holds*

$$\int_\Omega g w^2 \,\mathrm{d}x \leq 4 \int_\Omega (\nabla w)^2 \,\mathrm{d}x .$$

The proof of such lemmas can be found in [6].

Remark 1. If Ω_0 is star-shaped lemma 2 is a particular case of [7], lemma 1.

LEMMA 3. *For any* $\phi \in L^2(\Omega)$ *such that* $\int_\Omega \phi \,\mathrm{d}x = 0$, *there exists at least one function* $\varphi \in H$ *verifying*

$$(1) \qquad \begin{cases} \nabla \cdot \varphi = \phi & \text{in } \Omega , \\ | \nabla \varphi |_2 \leq c \, | \phi |_2 \end{cases}$$

with c positive constant depending only on the regularity of $\partial\Omega$.

The proof of such a lemma is given in [8].

Let us introduce, now, the two regularity classes where uniqueness will be proved.

$$\mathscr{I}_1 = \{(\rho, \boldsymbol{v}, \theta) \in [C_p^1(\bar{\Omega})]^6 \text{ and } (\rho, \boldsymbol{v}, \theta) \in L^2(\Omega) \times \mathbf{H} \times H \text{ such that } m_\theta < \theta < k'$$

$$|-\nabla \cdot (\boldsymbol{v}/2\rho) + \boldsymbol{v} \cdot \nabla\rho^{-1}| < k' k_1', \; |\boldsymbol{v}| < k'; \quad |\nabla \log \rho|_3 < k_1',$$

$$\max\{|\rho|_3, |\boldsymbol{v}|_6, |\rho\boldsymbol{v}|_3, |\nabla \cdot \boldsymbol{v}|_{3/2}, |\rho\nabla \cdot \boldsymbol{v}|_{3/2}\} < k'\}$$

$$\mathscr{I}_2 = \{(\rho, \boldsymbol{v}, \theta) \in [C_p^1(\bar{\Omega})]^6 \quad \text{and} \quad (\rho, \boldsymbol{v}, \theta) \in L^2(\Omega) \times \mathbf{H} \times H \quad \text{such that}$$

$$m_\theta \leq \theta \leq k'', \; |-\nabla \cdot (\boldsymbol{v}/2\rho) + \boldsymbol{v} \cdot \nabla\rho^{-1}| < k'' k_1''; \; \text{ for } r = |x| > 1:$$

$$|\boldsymbol{v}/\rho|(x) < k'' r, \; 0 < \rho(x) \leq k'' r^{-2}, \; |\nabla \log \rho(x)| < k_1'' r^{-1},$$

$$|\boldsymbol{v}(x)| < k'' r^{-1}, \; |\nabla \boldsymbol{v}(x)| < k'' r^{-1}\}.$$

§ 3. By a_i', $i = 1, \cdots, 4$, b_i', $i = 1, 2, 3$, and γ' (resp. a_i'', b_i'', γ'') we denote the constants a_i, b_i, γ introduced in section 2 of note I when we replace k, k_1', ν, b with the quantities $k', k_1', c_0, b' = |f|_3$ (resp $k'', k_1'', 2, b'' = \sup(|f|\, r)$) respectively. We are, now, in a position to state the main theorem.

UNIQUENESS THEOREM. *Let* $f \in C^0(\bar{\Omega}) \cap L^3(\Omega)$ (*resp.* $f \in C^0(\bar{\Omega})$ *and* $\sup(|f|\, r) < +\infty$) *and the numbers* $a_i', b_i', R, M, Pr, k_1', \gamma'$ (*resp.* a_i'', b_i'', $R, M, Pr, k_1'', \gamma''$) *verify the following relations*

$$(2) \qquad \begin{cases} \gamma' M^2/m_\theta < \min\{1/8c, 1/6a_1'\} \\ R < \min\{1/6c_0 k'^2, 1/[a_3' + 8(2a_4' + a_3')(2b_2' + b_3')]\} \\ Pr < \min\{1/2 Rb_1', 1/8 b_3'\} \\ k_1'/m_\theta < 1/6(a_1' + a_2'). \end{cases}$$

(*resp. the analogue of* (2) *when the substitution* $a_i' \to a_i''$, $b_i' \to b_i''$, $k' \to k''$, $k_1' \to k_1''$, $c_0 \to 2$ *is made*). *Then there exists at most one solution* $(\rho, \boldsymbol{v}, \theta) \in \mathscr{I}_1$ (*resp.* $\mathscr{I}_2$) *to the problem* $\mathscr{P}$ (cf. *Note I.*)

Remark 2. It is interesting to note that, unlike the unsteady case [3, 4], where uniqueness is achieved provided the density is bounded below by a decreasing function of r, here we must require that ρ is bounded *above* by an analogous function of r.

Proof. The proof will be given *per absurdum*. The starting point is again equation (6) of Note I, namely

$$
(3)\quad \begin{cases}
R\hat{\rho}\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{u}+R[\rho\boldsymbol{u}+\rho'(\boldsymbol{v}+\boldsymbol{u})]\cdot\nabla\boldsymbol{v}-\Delta\boldsymbol{u}-(\vartheta-1)\nabla\nabla\cdot\boldsymbol{u}=RM^{-2}\nabla p'+R\rho'\boldsymbol{f}\\
RPr\,(c_V/c_p)\,\{\hat{\rho}\hat{\boldsymbol{v}}\cdot\nabla\theta'+[\rho\boldsymbol{u}+\rho'\hat{\boldsymbol{v}}]\cdot\nabla\theta\}-\Delta\theta'=-RPr\,(R^*/c_p)\cdot\\
\quad\cdot(\hat{p}\nabla\cdot\boldsymbol{u}+p'\,\nabla\cdot\boldsymbol{v})+Pr\,(\vartheta-1)\,[(\nabla\cdot\boldsymbol{u})^2+2\,\nabla\cdot\boldsymbol{v}\nabla\cdot\boldsymbol{u}+\\
\quad+2Pr\,[\mathbf{D}':\mathbf{D}'+2\mathbf{D}:\mathbf{D}']\\
\nabla\cdot(\rho\boldsymbol{u}+\rho'\,\hat{\boldsymbol{v}})=0\\
\boldsymbol{u}\,|_{\partial\Omega}=\theta'\,|_{\partial\Omega}=0\\
\displaystyle\int_\Omega \rho'\,dx-0\,.
\end{cases}
$$

Let us multiply relations $(3)_{1,2}$ by $\boldsymbol{u}$ and θ', respectively. Integrating by parts over Ω and taking into account $(3)_{3,4}$, we deduce

$$
(4)\quad \begin{cases}
|\nabla\boldsymbol{u}|_2^2+(\vartheta-1)\,|\nabla\cdot\boldsymbol{u}|_2^2=(R/M^2)\displaystyle\int_\Omega \theta\rho'\nabla\cdot\boldsymbol{u}\,dx+F_1\\
|\nabla\theta'|_2^2=F_2
\end{cases}
$$

where F_i, $i=1,2$, are given in Note I. Notice that the summability hypothesis made on elements of $\mathscr{I}_1$ (resp. $\mathscr{I}_2$) is sufficient to ensure the finiteness of all integrals above. Multiplying by $\boldsymbol{\varphi}\in\mathbf{H}$ equation $(3)_1$ and integrating over Ω we have

$$
(5)\quad (R/M^2)\int_\Omega \theta\rho'\,\nabla\cdot\boldsymbol{\varphi}\,dx=-(R/M^2)\int_\Omega \hat{\rho}\theta'\nabla\cdot\boldsymbol{\varphi}\,dx-R\int_\Omega \rho'\boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx+
$$
$$
+\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{\varphi}\,dx+(\vartheta-1)\int_\Omega \nabla\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{\varphi}\,dx+R\int_\Omega \{(\hat{\rho}\boldsymbol{u}+\rho'\,\boldsymbol{v})\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{v}+
$$
$$
+\hat{\rho}\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}\}\,dx\,.
$$

Now, we let $\nabla\cdot\boldsymbol{\varphi}$ varying in $\mathbf{L}^2(\Omega)$ with $|\nabla\cdot\boldsymbol{\varphi}|_2=1$, by lemma 3 and lemma 1 (resp. lemma 2) and hypothesis $(2)_1$ we deduce as in Note I.

$$
(6)\quad |\rho'|_2\le(8k'c_0/7\,m_\theta)\,|\nabla\theta'|_2+(8\,M^2/7\,Rm_\theta)\,(\vartheta-1)\,|\nabla\cdot\boldsymbol{u}|_2+
$$
$$
+(8\,M^2c/7\,Rm_\theta)\,[1+2\,Rc_0\,k'^2]\,|\nabla\boldsymbol{u}|_2
$$

(resp. an analogous relation is valid in $\mathscr{I}_2$ when we make the replacement $k'\to k''$, $c_0\to 2$). Let us study, now, equations (4). To this end, employing

$(3)_3$, Holder (resp. Schwarz) inequality and lemma 1 (resp. lemma 2), we deduce the following relations

$$\left\{\begin{aligned}
&\int_\Omega \theta\rho'\nabla\cdot\boldsymbol{u}\,\mathrm{d}x \le k'\,k_1'\,|\,\rho'\,|_2\,(c_0\,|\,\nabla\boldsymbol{u}\,|_2 + |\,\rho'\,|_2)\\
&\mathrm{F}_1 \le \mathrm{R}\,(4\,b' + k'^2)\,|\,\rho'\,|_2\,|\,\nabla\boldsymbol{u}\,|_2 + \mathrm{R}k'^2\,c_0\,|\,\nabla\boldsymbol{u}\,|_2^2 + \frac{\mathrm{R}k'}{\mathrm{M}^2/c_0}\,|\,\nabla\theta'\,|_2\,\nabla\cdot\boldsymbol{u}\,|_2\\
&\mathrm{F}_2 \le \mathrm{P}r\left(\frac{\mathrm{R}k'^2}{c_p/c_{\mathrm{V}}} + 4\,k'\right)c_0\,|\,\nabla\boldsymbol{u}\,|_2\,|\,\nabla\theta'\,|_2 + \mathrm{P}r\,c_0\left(\frac{\mathrm{R}k'^2}{c_p/\mathrm{R}^*} + 2k'\,(\vartheta - 1)\right)\cdot\\
&\quad\cdot|\,\nabla\cdot\boldsymbol{u}\,|_2\,|\,\nabla\theta'\,|_2 + \mathrm{RP}r\,k'^2\left(\frac{c_{\mathrm{V}} + \mathrm{R}^*\,c_0}{c_p}\right)|\,\rho'\,|_2\,|\,\nabla\theta'\,|_2 + \mathrm{RP}r\,k'^2\,c_0^2\,\mathrm{R}^*\,\cdot\\
&\quad\cdot|\,\nabla\theta'\,|_2^2/c_p + \mathrm{P}r\,2\,k'\,|\,\nabla\boldsymbol{u}\,|_2^2 + \mathrm{P}r\,k'\,(\vartheta - 1)\,|\,\nabla\cdot\boldsymbol{u}\,|_2^2\,.
\end{aligned}\right.$$

Substituting these relations in (4), using inequality $|\,\nabla\cdot\boldsymbol{u}\,|^2 \le 3\,|\,\nabla\boldsymbol{u}\,|^2$, and employing (6) and (2) we deduce (as in Note I)

$$\left\{\begin{aligned}
&\left(\frac{5}{6} - \frac{\mathrm{M}^2\,a_1'}{m_\theta} - \frac{k_1'\,a_2'}{m_\theta}\right)|\,\nabla\boldsymbol{u}\,|_2^2 \le \mathrm{R}a_3'\,|\,\nabla\boldsymbol{u}\,|_2\,|\,\nabla\theta'\,|_2 + \mathrm{R}a_4'\,|\,\nabla\theta'\,|_2^2\\
&(1 - \mathrm{RP}r\,b_1')\,|\,\nabla\theta'\,|_2^2 \le \mathrm{P}r\,b_2'\,|\,\nabla\boldsymbol{u}\,|_2^2 + \mathrm{P}r\,b_3'\,|\,\nabla\boldsymbol{u}\,|_2\,|\nabla\theta'\,|_2\,.
\end{aligned}\right.$$

Such latter inequalities are just the analogue of (14) in Note I, when substitutions $a_i \to a_i'$, $b_i \to b_i'$ are made. Consequently, we obtain uniqueness in the same way as in Note I, in the case of a bounded domain.

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