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On the Singularities of the Newtonian two dimensional N-body Problem

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Meccanica. — *On the Singularities of the Newtonian two dimensional* N-*body Problem.* Nota (*) di CARLO MARCHIORO (**), (***) e MARIO PULVIRENTI (***), presentata dal Socio D. GRAFFI.

RIASSUNTO. — Si considera un sistema bidimensionale di N particelle interagenti tramite un potenziale di Newton o di Coulomb e si mostra che l'insieme delle condizioni iniziali che in un tempo finito possono condurre a delle singolarità possiede misura di Lebesgue nulla.

1. INTRODUCTION

One of the most important problems concerning the Newtonian dynamics of a N-body system is the analysis of its singularities. In particular it is very natural to ask whether the set of initial points of the phase space leading to a collapse in a finite time is or is not of Lebesgue measure zero.

This problem has been widely investigated, but it has not been completely solved. What is actually known is the following: " collisions ", i.e. the simultaneous presence of two or more particles at the same place of the physical space, are exceptional (in the sense of the Lebesgue measure). Nothing is known about other kinds of singularities, namely the system could become unbounded in a finite time.

The particular case when $N \leq 4$ has been completely solved. We address the reader to Ref. [1] and references quoted therein for a more complete insight on the subject.

In this note we give a short proof of the exceptionality of any kind of singularities for two dimensional systems. Our basic hypothesis is that the potential diverges logarithmically at short distance and, unfortunately our techniques do not apply in the more physically interesting three dimensional case.

Our approach works for Newtonian, as well as for Coulomb systems, and is based on ideas similar to those used for vortex motion (See Ref. [2] and [3]) and for a class of vector fields with a compact set of singularities [4].

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2. RESULTS AND PROOFS

We consider a N-particle mechanical system in $\mathbf{R}^2$, described by the following Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \gamma_{ij} \ln |x_i - x_j| . \tag{2.1}$$

Here, as usual, x_i, p_i and m_i denote position, momentum and mass of the i-th particle, and γ_{ij} are numbers related to the nature of the system. For example $\gamma_{ij} = m_i m_j$ in the case of gravitational forces and $\gamma_{ij} = -q_i q_j$ with q_i the electric charge of the i-th particle, in the case of Coulomb systems.

We introduce a regularized version of the Hamiltonian 2.1 by defining

$$H_\varepsilon = \sum_{i=1}^{N} \frac{p_i^2}{2\, m_i} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \gamma_{ij} \ln_\varepsilon |x_i - x_j| \tag{2.2}$$

where $\ln_\varepsilon |\cdot| : \mathbf{R} \to \mathbf{R}$ is a C^∞ even function, satisfying

$$\ln_\varepsilon |t| = \ln |t| \quad \text{if} \quad |t| \geq \varepsilon \quad , \quad 0 < \varepsilon < 1 \quad \min_{|t| < \varepsilon} \ln_\varepsilon |t| = 2 \ln \varepsilon . \tag{2.3}$$

We denote by S_t^ε the (smooth) Hamiltonian flow generated by the Hamilton function (2.2) i.e.

$$S_t^\varepsilon X = (x_1^\varepsilon(t), p_1^\varepsilon(t) \ldots x_N^\varepsilon(t), p_N^\varepsilon(t)) , \qquad t \in \mathbf{R} \tag{2.4}$$

is the orbit in the phase space of the phase point which is $X = x_1 p_1 \ldots x_N p_N$ at time zero.

Let A be a bounded, measurable set in the phase space $\mathbf{R}^{2N} \times \mathbf{R}^{2N}$ of diameter R and $\lambda(dX) = \dfrac{dx_1\, dp_1 \ldots dx_N\, dp_N}{\text{meas A}}$ is the Lebesgue measure normalized to one when restricted on A.

The main result of this note is the following.

THEOREM 2.1. *Consider the following set*

$$B_T(\varepsilon) = \{X \in A \mid \min_{\substack{i,j \\ i \neq j}} \inf_{0 \leq t \leq T} |x_i^\varepsilon(t) - x_j^\varepsilon(t)| \leq \varepsilon\} , \qquad \varepsilon < 1 . \tag{2.5}$$

Then, for all $\xi \in (0, 1)$, *there exists a positive constant* C, *depending on* ξ, T, A *and* N *but independent of* ε, *such that*

$$\lambda(B_T(\varepsilon)) \leq c\, \varepsilon^\xi . \tag{2.6}$$

Proof. Let us define the following function

$$\varphi(X) = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{|x_i - x_j|^{\alpha}} \qquad \alpha \in (0, 1). \tag{2.7}$$

We have

$$\frac{d}{dt} \varphi(S_t^{\varepsilon} X) = -\alpha \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{(x_i^{\varepsilon}(t) - x_j^{\varepsilon}(t)) \cdot (\dot{x}_i^{\varepsilon}(t) - \dot{x}_j^{\varepsilon}(t))}{|x_i^{\varepsilon}(t) - x_j^{\varepsilon}(t)|^{\alpha+2}} \tag{2.8}$$

and hence

$$\left| \frac{d}{dt} \varphi(S_t^{\varepsilon} X) \right| \leq \alpha \sum_{\substack{i,j=1 \\ j \neq j}}^{N} \frac{|\dot{x}_i^{\varepsilon}(t)| + |\dot{x}_j^{\varepsilon}(t)|}{|x_i^{\varepsilon}(t) - x_j^{\varepsilon}(t)|^{\alpha+1}}. \tag{2.9}$$

We define

$$P = \max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} |p_i^{\varepsilon}(t)| \quad \text{and} \quad m = \max_i m_i, \ \gamma = \max_{i,j} |\gamma_{ij}|.$$

Because of the conservation of the energy we have

$$|P_i^{\varepsilon}(t)| \leq \sqrt{2 m_i \sum_{j=1}^{N} \frac{p_j^{\varepsilon}(t)^2}{2 m_j}} \leq \sqrt{2 m_i |H_{\varepsilon}(X)|} + \sqrt{N^2 m \gamma \ln \varepsilon^{-1}} + \tag{2.10}$$

$$+ \sqrt{N^2 m_i \gamma \ln_{\varepsilon} 2 [R + TP]}$$

hence for ε sufficiently small

$$P \leq C_0 \sqrt{\ln \varepsilon^{-1}}. \tag{2.11}$$

From now on C_i, $i = 0, 1 \ldots$ will denote positive constants depending on physical parameters and A and T but not on ε.

Finally we have

$$\left| \frac{d}{dt} \varphi(S_t^{\varepsilon} X) \right| \leq C_1 \sqrt{\ln \varepsilon^{-1}} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{|x_i^{\varepsilon}(t) - x_j^{\varepsilon}(t)|^{\alpha-1}}. \tag{2.12}$$

Moreover by Fubini Theorem and conservation of the measure

$$(2.13)\qquad \int_A \sup_{0\le t\le T} \varphi(S_t^\varepsilon X)\, d\lambda(X) \le \int_A d\lambda(X)\,\varphi(x) + \int_A d\lambda(X) \sup_{0\le t\le T}$$

$$\int_0^t |\dot\varphi(S_\tau^\varepsilon X)|\, d\tau \le C_2 + C_1 \sqrt{\ln \varepsilon^{-1}} \int_0^T d\tau \int_A d\lambda(X) \sum_{\substack{i,j=1\\ i\neq j}}^N \frac{1}{|x_i^\varepsilon(\tau) - x_j^\varepsilon(\tau)|^{\alpha+1}} \le$$

$$\le C_2 + C_1 \sqrt{\ln \varepsilon^{-1}} \int_0^T d\tau \int d\lambda(X)\, \chi_A(S_{-\tau}^\varepsilon(X)) \sum_{i\neq j} \frac{1}{|x_i - x_j|^{\alpha+1}}$$

where χ_A denoted, as usual, the indicator of the set A.

The diameter of the region $S_t^\varepsilon A = \{Y \mid Y = S_t^\varepsilon X, X \in A\}$ is certainly bounded by $C_3 \sqrt{\ln \varepsilon^{-1}}$ (ε small) and hence

$$(2.14)\qquad \int d\lambda(X)\, \chi_A(S_{-\tau}^\varepsilon X) \sum_{\substack{i,j=1\\ i\neq j}}^N \frac{1}{|x_i - x_j|^{\alpha+1}} \le \frac{1}{\text{meas } A} \sum_{\substack{i,j=1\\ i\neq j}}^N$$

$$\int_{\substack{|x_i| \le C_3 \sqrt{\ln \varepsilon^{-1}}\\ |p_i| \le C_3 \sqrt{\ln \varepsilon^{-1}}}} \frac{1}{|x_i - x_j|^{\alpha+1}} \prod_{i=1}^N dx_i\, dp_i \le C_4 (\ln \varepsilon^{-})^{2N}$$

Finally, by Tchebychev inequality

$$(2.15)\qquad \lambda(B_T(\varepsilon)) \le \lambda(\{X \in A \mid \sup_{0\le t\le T} |\varphi(S_t^\varepsilon X)| > \varepsilon^{-\alpha}\}) \le$$

$$\le \left(\int \lambda(dX) \sup_{0\le t\le T} |\varphi(S_t^\varepsilon X)|\right) \varepsilon^\alpha \le C_4\, \varepsilon^\alpha (\ln \varepsilon^{-1})^{2N}. \quad \blacksquare$$

We are now able to prove the absence of singularities for a full measure set of initial conditions in A.

THEOREM 2.2. *For any initial condition* $X \in \mathbf{R}^{2N} \times \mathbf{R}^{2N}$, *but for a set of Lebesgue measure zero, a smooth hamiltonian flow* $S_t X$ *generated by the Hamiltonian function* (2.1) *can be defined. Moreover*

$$S_t X = \lim_{\varepsilon\to 0} S_t^\varepsilon X .$$

Proof. Let T be a fixed arbitrary time. We have $\mathbf{R}^{2N} \times \mathbf{R}^{2N} = \bigcup_{k=1}^{\infty} \Sigma_k$ where Σ_k denotes the 4 N dimensional sphere of radius K. For $X \in \Sigma_k$ we can apply Theorem 2.1 replacing A by Σ_k.

Then, there exists a set N_k such that $\lambda(N_k) = 0$ and, for all $X \in \Sigma_k \setminus N_k$, there exists $\varepsilon > 0$ for which $S_t^{\varepsilon} X = S_t X$, $t \leq T$. From this the thesis easily follows.

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