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Geodesia. — *The current situation in the linear problem of Molodenskii.* Nota I (*) di FAUSTO SACERDOTE e FERNANDO SANSÒ, presentata dal Socio L. SOLAINI.

RIASSUNTO. — Si studiano le condizioni per l'esistenza, l'unicità e la stabilità della soluzione debole del problema lineare di Molodenskii in approssimazione quasi-sferica, generalizzando una tecnica perturbativa usata in precedenza per la soluzione di tipo classico.

La procedura seguita richiede delle condizioni di maggior regolarità per il contorno, di quelle usate nell'analisi del problema « semplice ». Il risultato ottenuto è l'esistenza e unicità di una soluzione con derivate seconde a quadrato integrabile, se il bordo ammette curvatura limitata.

1. INTRODUCTION

The problem of Molodenskii is the basic boundary value problem of physical geodesy and consists in searching for the figure of the earth (the unknown boundary S) and for the external gravity potential $w(x)$, given the gravity potential itself w and the gravity field $\boldsymbol{g} = \nabla w$ on the unknown S: $w|_S, \boldsymbol{g}|_S$. The gravity potential is assumed to be split into the gravitational part and the centrifugal part, the former being the Newtonian potential of a static mass distribution, the latter being the single term $\frac{1}{2}\omega^2(x^2+y^2)$ corresponding to the hypothesis of a rigid uniformly rotating planet. As such, Molodenskii's problem is a rather difficult non linear, free boundary value problem for the Laplace operator. It has been treated in the classical formulation by Hörmander [1] and, after a suitable Legendre transformation, in a new formulation, known in geodesy as the gravity space approach, by Sansò [5] and Witsch [8].

The analysis, in the classical mathematical sense, of this problem is of interest to geodesists since we would like to know whether the solution exists, is unique and specially which are the regularity properties of this solution for a given regularity of the data. It is this last point, namely the continuous dependence of the solution on the data, which is of special importance since it is the basis for evaluating various approximation methods proposed on geodesy.

In this sense, for the physical reason that the surface of the earth is generally

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a rough surface, which at most can be assumed to satisfy a cone condition, we would consider as satisfactory a theorem of existence, uniqueness and continuous dependence which requires no more than the boundedness of the first derivatives of the boundary data. This result has not yet been achieved, even for the linearized problem of Molodenskii. The reason is that, when linearized, the geodetic boundary value problem appears as an oblique derivative problem for the Laplace equation (cfr. Krarup [2], Hörmander [1], Sansò [6]) on a boundary, the so called telluroid S_0, which is as rough as the true surface of the earth. In this case the unknown function becomes the anomalous potential T defined as the difference between the actual potential and some " normal " reference potential, $T = w - w_0$: this is easily seen to be a harmonic function. Once the potential T is found by solving the oblique derivative problem, the vector ξ describing the displacement between the actual surface and the approximated telluroid can be recovered: the vector ξ depends on ∇T computed on the telluroid S_0. We conclude that any reasonable solution of our problem should be so regular as to admit the trace of ∇T on the boundary, and this gradient must be at least an L^2 function.

However, in mathematical literature the oblique derivative problem is usually treated, even in the weak sense, with milder assumptions as to the shape of the boundary. Moreover the classical weak solution for such problems is H^1, so that the trace of ∇T at the boundary is not defined.

Subsequently a specific analysis of linear Molodenskii's problem was begun, to see whether we could cope with the above requirements.

In the next section we first recall the results obtained in Sansò [7], concerning the so called simple Molodenskii's problem, where a simple reference potential $w_0 = a/r$ is assumed; subsequently we state the main result of this paper, which is proved in section 3.

2. General approach

In Sansò [6] the existence and uniqueness of a classical solution $T \in C_{2+\varepsilon}(\Omega)$ of the simple problem of Molodenskii

$$(2.1) \qquad \begin{cases} \Delta T = 0 & \text{in } \Omega \\ \dfrac{1}{2} r \dfrac{\partial T}{\partial r} + T \,|_{\partial\Omega} = u + \boldsymbol{a} \cdot \mathbf{A} \\ T = \dfrac{\alpha}{r} + 0\,(r^{-3}) \end{cases}$$

is proved for $u \in C_{1+\varepsilon}(\Omega)$; the solution can be written as $T = Gu$, where $G : C_{1+\varepsilon}(\partial\Omega) \to C_{2+\varepsilon}(\Omega)$ is a continuous operator. Here Ω is the unbounded domain exterior to the surface $\partial\Omega$; its complement in $\mathbf{R}^3$ is a $\mathcal{N}^{(0),1}$ star-shaped domain (1); $A_j = (R/r)^2 Y_{1j}|_{\partial\Omega}$, where R is such that the spherical surface centred at the origin with radius R is wholly contained in Ω.

This result is then used to prove the existence and uniqueness of the solution of the geodetic boundary value problem in almost spherical approximation, i.e. when the "isozenithal" field $\boldsymbol{m}_0$ involved in the boundary condition

$$-\boldsymbol{m}_0 \cdot \nabla T + T|_{\partial\Omega} = u + \boldsymbol{a} \cdot \mathbf{A} \tag{2.2}$$

is close enough to $-\frac{1}{2}\boldsymbol{r}$:

$$\boldsymbol{m}_0 = -\frac{1}{2}\boldsymbol{r} + \mu_0 \;, \quad \mu_0 \quad \text{"sufficiently" small in } C_{1+\varepsilon} \,. \tag{2.3}$$

The proof takes advantage of the fact that $\mu_0 \cdot \nabla T|_{\partial\Omega} \in C_{1+\varepsilon}(\partial\Omega)$; consequently, if the boundary condition is written as

$$\frac{1}{2} r \frac{\partial T}{\partial r} + T|_{\partial\Omega} = \mu_0 \cdot \nabla T|_{\partial\Omega} + u + \boldsymbol{a} \cdot \mathbf{A} \tag{2.4}$$

the solution T can be found as the fixed point of the transformation of $C_{2+\varepsilon}(\Omega)$ into itself defined by

$$T \to G(\mu_0 \cdot \nabla T|_{\partial\Omega} + u) \tag{2.5}$$

in fact the norm of this transformation is dominated by $\|\mu_0\|_{C_{1+\varepsilon}}$ and becomes less than 1 when this quantity is small enough. The aim of this paper is to seek for a generalization of the above procedure to the weak solutions; what turns out to be non-elementary. As a matter of fact, in Sansò [7] it is proved that, if $u \in H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique weak solution of (2.1) belonging to the space $HH'^1(\Omega)$ of harmonic functions in $L^2_{loc}(\Omega)$ vanishing at infinity, with zero first degree harmonic components and distributional first derivatives in $L^2(\Omega)$.

Moreover, by a suitable regularization theorem it is proved that $\nabla T|_{\partial\Omega}$ belongs to $L^2(\partial\Omega)$; however this result is not strong enough to be able to apply the fixed point theorem as in (2.5), since we would need for that $\nabla T|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$. What we need is therefore a stronger regularization theorem; to this aim we must make stronger assumptions on the regularity

(1) We recall that, following Nečas [4], pag. 55, a bounded domain is of class $\mathcal{N}^{(k),\mu}$ (k non-negative integer, $0 \leq \mu \leq 1$), if its boundary can be represented by a system of local charts, defined by functions that are μ-Hölder together with their derivatives of order $\leq k$.

of the boundary. The result we shall prove in next section is the following theorem:

If $\Omega \in \mathcal{N}^{(1),1}$, *then the solution* T *of problem* (2.1) *has distributional second derivatives belonging to* $L^2(\partial\Omega)$; ∇T *belongs to* $H^1_{loc}(\Omega)$, *so that* ∇T *has a trace on* $\partial\Omega$ *in* $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$.

As an immediate consequence, we can state that, since the field μ_0 defined in (2.3) is of class C^1, $\mu_0 \cdot \nabla T \in H^1_{loc}(\Omega)$ and its trace belongs to $H^{\frac{1}{2}}(\partial\Omega)$. Hence the procedure outlined in the first part of this section (cfr. (2.4), (2.5)) can be applied to solve the boundary value problem in almost spherical approximation, provided that $\|\mu_0\|_{C^1}$ is small enough. In fact, let us define $HH'^2(\Omega)$ as the space of functions in $HH'^1(\Omega)$ with second distributional derivatives in $L^2(\Omega)$, with

$$\|u\|_{HH'^2(\Omega)} = \|u\|_{HH'^1(\Omega)} + \sum_{jk} \left\| \frac{\partial^2 u}{\partial x_j \, \partial x_k} \right\|_{L^2(\Omega)} .$$

Then the solution T belongs to $HH'^2(\Omega)$ and we can introduce the operator G as in (2.5), with $G : H^{\frac{1}{2}}(\partial\Omega) \to HH'^2(\Omega)$.

Now, let us consider the operator $B = G\gamma\,\mu_0 \cdot \nabla$ (where γ is the trace operator); $B : HH'^2(\Omega) \to HH'^2(\Omega)$. If the components of μ_0 and their first derivatives have a sufficiently small maximum on $\partial\Omega$, the norm of B can be made small enough to be able to apply the fixed point theorem. In this way we find a weak solution of the problem in almost spherical approximation that belongs to $HH'^2(\Omega)$, and our problem has a solution with the required regularity conditions.

In a following note we shall prove by a different and more direct approach that, even if we require only $\Omega \in \mathcal{N}^{(0),1}$, which is more natural in geodetic problems, a weak solution can be found; in this case, however, we can prove only a weaker regularity result, i.e. $\nabla T\,|_{\partial\Omega} \in L^2(\Omega)$.

3. Solution in a domain with regular boundary

The goal of this section is to prove the theorem previously stated.

We recall that in [7] the solution T of (2.1) is found by extending to a harmonic function in the whole Ω the function u given on the boundary, and then by proving that the operator B^{-1}, where $B = \frac{1}{2} r \frac{\partial}{\partial r} - I$, establishes a one-to-one map of the space $HH'^1(\Omega)$ into itself. Hence

$$(3.1) \qquad u \in HH'^1(\Omega) \Rightarrow T \in HH'^1(\Omega) \Rightarrow \frac{\partial T}{\partial r} = \frac{2}{r}(u - T) \in HH'^1(\Omega) \subset H^1_{loc}(\Omega) .$$

What remains to be proved is then that the non-radial components of the gradient of T belong to $H^1_{loc}(\Omega)$ too. Let $r = R(\theta, \lambda)$ represent the boundary $\partial\Omega$; $R(\theta, \lambda)$ is in $C^{1,1}$ in agreement with our assumption that $\partial\Omega \in \mathcal{N}^{(1),1}$. In Ω we can introduce coordinates t, θ, λ in the following way

$$(3.2) \qquad \begin{cases} x = tR(\theta, \lambda) & \sin\theta\cos\lambda \\ y = tR(\theta, \lambda) & \sin\theta\sin\lambda \\ z = tR(\theta, \lambda) & \cos\theta . \end{cases} \qquad 1 \leq t < \infty$$

This transformation is not one-to-one along the polar axis; since we are interested only in local considerations, we can overcome this difficulty for example by dividing our domain into two subdomains and by using suitably rotated coordinates.

For simplicity of notation we rename x, y, z by x_1, x_2, x_3 and t, θ, λ by t_1, t_2, t_3. The transformation can be simply denoted as $\mathbf{x} = \mathbf{F}(\mathbf{t})$. From the preceding remark we can assume that it is invertible, with Jacobian J_F bounded, $\min |J_F| > 0$; we denote by $\mathbf{t} = \mathbf{G}(\mathbf{x})$ the inverse transformation. Moreover, $\mathbf{G}$ is of class $C^{1,1}$ and, at least almost everywhere, we have

$$(3.3\,a) \qquad \frac{\partial^2 w}{\partial x_j\, \partial x_k} = \sum_l \frac{\partial^2 G_l}{\partial x_j\, \partial x_k} \frac{\partial \hat{w}}{\partial t_l} + \sum_{i,l} \frac{\partial G_i}{\partial x_j} \frac{\partial G_l}{\partial x_k} \frac{\partial^2 \hat{w}}{\partial t_i\, \partial t_l} \Rightarrow$$

$$(3.3\,b) \qquad \Rightarrow \Delta W = \sum_l \Delta G_l \frac{\partial \hat{w}}{\partial t_l} + \sum_{i,l} \nabla G_i \cdot \nabla G_l \frac{\partial^2 \hat{w}}{\partial t_i\, \partial t_l} \equiv \hat{\Delta} \hat{w}$$

where $\hat{w}(\mathbf{t}) = w(\mathbf{F}(\mathbf{t}))$.

Owing to the specific form of (3.2), the following properties can easily be verified:

i) $J_F(\mathbf{t}) = t_1^2 \cdot J(t_2, t_3)$

ii) $\dfrac{\partial t_1}{\partial x_j}$, $j = 1, 2, 3$, expressed as function of $\mathbf{t}$, are independent of t_1.

iii) $\dfrac{\partial t_i}{\partial x_j}$, $i = 2, 3$, $j = 1, 2, 3$, can be written $1/t_1$ multiplied by functions of t_2, t_3.

(3.4) iv) $\dfrac{\partial^2 t_1}{\partial x_j\, \partial x_k}$ $j, k = 1, 2, 3$, can be written as $1/t_1$ multiplied by functions of t_2, t_3.

v) $\dfrac{\partial^2 t_i}{\partial x_j\, \partial x_k}$ $i = 2, 3$, $j, k = 1, 2, 3$, can be written as $1/t_1^2$ multiplied by functions of t_2, t_3

vi) $\dfrac{\partial}{\partial t_1} = R(\theta, \lambda) \dfrac{\partial}{\partial r}$.

Consequently (3.3a) can be written as

$$(3.5)\quad \frac{\partial^2 w}{\partial x_j\, \partial x_k} = \frac{\partial^2 \hat{w}}{\partial t_1^2}\, \alpha_{jk}(t_2, t_3) + \frac{1}{t_1}\left(\frac{\partial \hat{w}}{\partial t_1}\, \beta_{jk}(t_2, t_3) + \sum_{2}^{3}{}_i \frac{\partial^2 \hat{w}}{\partial t_1\, \partial t_i}\, \Phi_{jk}^{(i)}(t_2, t_3)\right) +$$

$$+ \frac{1}{t_1^2}\left(\sum_{2}^{3}{}_i \frac{\partial \hat{w}}{\partial t_i}\, \psi_{jk}^{(i)}(t_2, t_3) + \sum_{2}^{3}{}_{i,l} \frac{\partial^2 \hat{w}}{\partial t_i\, \partial t_l}\, \chi_{jk}^{(il)}(t_2, t_3)\right)$$

where all the coefficients of the derivatives are bounded with the first and second derivatives of **G**.

Now, consider the surface $\Sigma_\tau = \{t_1 = \tau, \tau > 1\}$, that is internal to Ω; as T is a harmonic function, bounded in every compact internal to Ω, it is regular on Σ_τ.

Let us denote by Δ_τ the part in the right hand side of (3.5) that does not contain derivatives with respect to t_1. Thus the Laplace operator $\hat{\Delta}$ is decomposed according to the formula

$$\hat{\Delta} = \mathrm{L} + \Delta_\tau \ ,$$

where L and Δ_τ are both linear second-order differential operators with bounded coefficients; L contains the first and second derivative with respect to t_1 and the mixed derivatives $\dfrac{\partial}{\partial t_i}\, \dfrac{\partial}{\partial t_1}$ $(i = 2, 3)$. We see that $\tau^2 \Delta_\tau$ does not depend on τ. Let us define $\hat{\mathrm{T}}(t) = \mathrm{T}(\mathbf{F}(t))$ and let $\hat{\mathrm{T}}_\tau(t_2, t_3)$ be the restriction of $\hat{\mathrm{T}}$ to Σ_τ. We shall prove that the following inequality holds:

$$(3.6)\qquad \|\hat{\mathrm{T}}_\tau\|^2_{\mathrm{H}^2(\mathrm{D})} \le c\,(\|\hat{\mathrm{T}}_\tau\|^2_{\mathrm{H}^1(\mathrm{D})} + \|\tau^2 \Delta_\tau \hat{\mathrm{T}}_\tau\|^2_{\mathrm{L}^2(\mathrm{D})})$$ (1)

where $\mathrm{D} \subset \mathbf{R}^2$ is the (bounded) domain of variation of (t_2, t_3); c is a costant independent of τ.

However what we really want to prove is that

$$(3.7)\qquad \int_\Omega \left|\frac{\partial^2 \mathrm{T}}{\partial x_j\, \partial x_k}\right|^2 \mathrm{d}^3 x < \infty \,.$$

Let us see first that (3.6), together with (3.1), implies (3.7).

We examine (3.5), with $w = \mathrm{T}$. We already know that the terms in the right hand side that contain first or second derivatives with respect to t_1 belong

(1) $$\|\hat{\mathrm{T}}\|^2_{\mathrm{H}^2(\mathrm{D})} = \int_{\mathrm{D}} \left\{|\hat{\mathrm{T}}|^2 + \sum_{2}^{3}{}_{i,l}\left(\frac{\partial^2 \hat{\mathrm{T}}}{\partial t_i\, \partial t_l}\right)^2\right\} \mathrm{d}t_2\, \mathrm{d}t_3 \,.$$

to $L^2(\Omega)$, according to (3.1) and to (3.4), vi). Hence we focus our attention on the last term. We have, owing to (3.6).

$$\int_\Omega \left| \frac{1}{t_1^2} \frac{\partial^2 T}{\partial t_i \, \partial t_l} \right|^2 d^3x = \int_1^\infty dt_1 \, t_1^2 \int_D \frac{1}{t_1^4} \left| \frac{\partial^2 \hat{T}}{\partial t_i \, \partial t_l} \right|^2 J(t_2, t_3) \, dt_2 \, dt_3 \leq \tag{3.8}$$

$$\leq c \max J \int_1^\infty \frac{dt_1}{t_1^2} \| \hat{T}_{t_1} \|^2_{H^1(D)} + \int_1^\infty t_1^2 \, dt_1 \int_D | \Delta_{t_1} \hat{T}_{t_1} |^2 \, dt_2 \, dt_3 \quad (i, l = 2, 3) \, .$$

The first term is easily seen to be bounded by $k \| T \|_{HH'^1(\Omega)}$. As for the second term, we recall that, being T a harmonic function, $\Delta_{t_1} \hat{T}_{t_1} = - L\hat{T}$; as all terms in L contain derivatives with respect to t_1, $L\hat{T}$ belongs to $L^2(\Omega)$ according to (3.1), and $\| L\hat{T} \|_{L^2(\Omega)}$ is bounded by $\bar{k} \| u \|_{HH'^1(\Omega)}$ (see (3.1)). We have

$$\int_1^\infty t_1^2 \, dt_1 \int_D | \Delta_{t_1} \hat{T}_{t_1} |^2 \, dt_2 \, dt_3 \leq \frac{1}{\min J} \| L\hat{T} \|^2_{L^2(\Omega)} \, . \tag{3.9}$$

Summing up we can conclude that T is in $HH'^2(\Omega)$ and its norm is bounded by $\| u \|_{HH'(\Omega)}$.

Now we come to the proof of (3.6), that relies on Theorem 3.1, Chap. 2 of Lions-Magenes [3], vol. 1. There, an inequality like (3.6) is established for a function ψ defined in $\mathbf{R}^m$ with support in a sufficiently small ball.

As we need to set up a finite covering of the set D, we must prove that in our case we can choose the radius ρ of the ball and determine the constant c independently of the center of the ball in $D \subset \mathbf{R}^2$. From the proof given in [3], we see that ρ and c are determined by the oscillation of the coefficients of the second derivatives, which by (3.3a) are products of the first derivatives of **G**, and by the magnitude of the coefficients of the first derivatives, which are second derivatives of **G**. Here the first and second derivatives of **G** are bounded on D, so that our statement is certainly true.

We introduce now a partition of unity $\{\psi_i\}$ corresponding to the finite covering of D.

We have $\hat{T}_\tau = \sum_i (\psi_i \hat{T}_\tau)$ where the summation is finite and for any $\psi_i \hat{T}_\tau$ inequality (3.6) holds.

$$\| \hat{T} \|^2_{H^2(D)} \leq k \Sigma \| \psi_i \hat{T}_\tau \|^2_{H^2(D)} \leq kc \Sigma \left(\| \psi_i \hat{T}_\tau \|^2_{H^1(D)} + \right. \tag{3.10}$$

$$\left. + \| \tau^2 \Delta_\tau (\psi_i \hat{T}) \|^2_{L^2(D)} \right) .$$

As ψ_i are bounded with their first and second derivatives, we can write

$$\| \psi_i \hat{T}_r \|^2_{H^1(D)} \leq c_1 \| \hat{T}_\tau \|^2_{H^1(D)} \; ; \tag{3.11}$$
$$\| \Delta_\tau (\psi_i \hat{T}_\tau) \|^2_{L^2(D)} \leq c_2 (\| \Delta_\tau \hat{T}_\tau \|^2_{L^2(D)} + \| \hat{T}_\tau \|^2_{H^1(D)}) \, .$$

Introducing (3.11) into (3.10), (3.6) is easily obtained (with a constant $\bar{c}$ which is obviously different from c in (3.10)).

N.B. — We remark that, in performing our computations, we have repeatedly used the boundedness of the second derivatives of $\mathbf{R}(\theta , \Phi)$ in (3.2). Hence Ω must be at least in $\mathcal{N}(^1)^{,1}$. Of course we can use, instead of (3.2), a transformation that is more regular inside Ω; but, if $\partial\Omega$ is not regular enough, second derivatives are not bounded when we approach $\partial\Omega$.

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