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On weak i-homotopy equivalences of modules

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Geometria. — *On weak* i*–homotopy equivalences of modules.* Nota di HE ZHENG–XU, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Si definisce il gruppo di *i*–omotopia di un singolo modulo e si introduce la nozione di equivalenza *i*–omotopica *debole*. Sotto determinate condizioni per l'anello di base Λ oppure per i moduli considerati, le equivalenze *i*–omotopiche deboli coincidono con le equivalenze *i*–omotopiche (forti).

The homotopy equivalence is a basic notion in the homotopy theory. In the case when the objects are modules, we have a very strong condition for homotopy equivalences; that is: a map of modules $\Phi : A \to B$ is an *i*–homotopy equivalence if and only if Φ can factored into:

$$A \to A \oplus U \xrightarrow{\Phi'} B \oplus V \to B$$

with U, V injective modules and Φ' an isomorphism of modules (see [4, Th. 13.7], also [2] for a similar result for pairs of modules; we will use the notations from [4, ch. 13]).

In this paper, we introduce the notion of *weak* *i*–homotopy equivalence; we show that under some conditions the weak *i*–homotopy equivalences are the same as the (strong) *i*–homotopy equivalences. Incidentally, we will define the *i*–homotopy groups of a *single* module.

Let Λ be a (fixed) commutative unitary ring. Let $\mathscr{I}$ be the family of all ideals of Λ, any element of $\mathscr{I}$ may be considered as a Λ–module. For a module (i.e. Λ–module) A, denote

$$\bar{\pi}_n(A) = \coprod_{I \in \mathscr{I}} \bar{\pi}_n(I, A) \qquad (n \geq 0),$$

and we call $\bar{\pi}_n(A)$ the *n–th i–homotopy group of the module* A. Clearly, any map $\Phi : A \to B$ induces homomorphisms of *i*–homotopy groups $\Phi_* : \bar{\pi}_n(A) \to \bar{\pi}_n(B)$. Similarly, for any pair α, define

$$\bar{\pi}_n(\alpha) = \coprod_{I \in \mathscr{I}} \bar{\pi}_n(I, \alpha) \qquad (n \geq 1);$$

and any map of pairs induces homomorphisms of such groups.

(*) Nella seduta del 14 gennaio 1984.

From [4, Theorem 13.15] we deduce:

PROPOSITION 1. *For any* $\alpha : A \to B$, *we have an exact sequence*:
$\cdots \to \pi_n(A) \to \overline{\pi}_n(B) \to \overline{\pi}_n(\alpha) \to \overline{\pi}_{n-1}(A) \to \cdots \to \overline{\pi}(A) \to \overline{\pi}(B)$.
Moreover if α *is a fibre map with the fibre* F, *then we get the following exact sequence*:

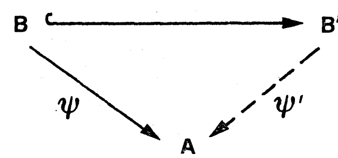
$$\cdots \to \overline{\pi}_n(A) \to \overline{\pi}_n(B) \to \overline{\pi}_{n-1}(F) \to \overline{\pi}_{n-1}(A) \to \cdots$$

DEFINITION. A map of modules $\Phi : A \to B$ is called a *weak i-homotopy equivalence* if $\Phi_* : \overline{\pi}_n(A) \to \overline{\pi}_n(B)$ is isomorphic for any $n \geq 0$.

Of course, any *i*–homotopy equivalence is also a weak *i*–homotopy equivalence. The following proposition justifies the above definition.

PROPOSITION 2. $A \simeq_i 0$ *if and only if* A *is weakly i-homotopy equivalent with* 0.

Proof. The « only if » part is trivial. As for the « if » part, we prove a stronger result: $\overline{\pi}(A) = 0$ implies that A is injective (i.e. $A \simeq_i 0$). Let $\psi : B \to A$ and let $B \subset B'$:



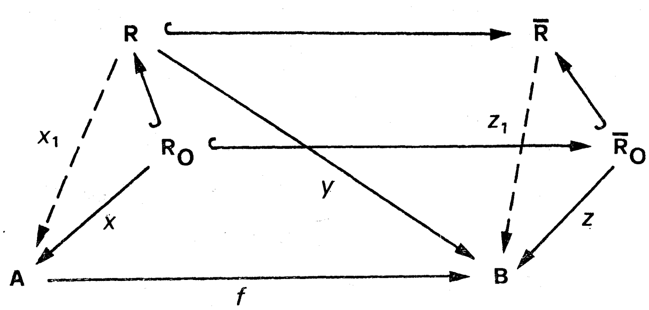
We need the existence of some $\psi' : B' \to A$ satisfying $\psi'/B = \psi$, which can be deduced (using Zorn's Lemma) from the fact that $\overline{\pi}(I, A) = 0$ for any $I \in \mathcal{J}$.

Sometimes we require the ring Λ to have the following property:

(*) For any $I \in \mathcal{J}$, $SI \simeq_i 0$.

Observe that (*) implies $\overline{\pi}_n(A) = 0$ for $n \geq 1$. Any hereditaryr ing satisfies (*).

LEMMA 1. *Assume that* Λ *satisfies* (*). *Let* $f : A \to B$ *be a weak i–homotopy equivalence, let* R_0 *be a submodule of a* Λ*–module* R *and assume that there exists* $a \in R - R_0$ *such that* $R = R_0 + \Lambda a$. *For any* $y : R \to B$, $x : R_0 \to A$ *and* $z : \overline{R}_0 \to B$ *with* $f \circ x + z/R_0 = y/R_0$, *there exist* $x_1 : R \to A$ *and* $z_1 : \overline{R} \to B$ ($\overline{R}$ *is chosen to include* $\overline{R}_0$) *such that* $x/R_0 = x$, $z_1/R_0 = z$, $fox_1 + z_1/R = y$:



Proof. Let $I = \{\lambda \in \Lambda ; \lambda a \in \bar{R}_0\}$, define $u : I \to A$, $v : \Lambda \to B$ and $w : I \to B$ by $u(\lambda) = x(\lambda a)$, $v(\lambda) = y(\lambda a)$ and $w(\lambda) = z(\lambda a)$. Let $\bar{R}_1 = (\bar{R}_0 \oplus \bar{\Lambda}) / \{(\lambda a, -\lambda); \lambda \in I\}$. $\bar{R}_1 \simeq_i SI$, so $\bar{R}_1$ is injective by (*). Obviously $\bar{R}_0 \subset \bar{R}_1$. The map $i : R \to \bar{R}_1$ defined by $i(a_0 + \lambda a) = [a_0, \lambda]$ ($a_0 \in R_0$, $\lambda \in \Lambda$) is an inclusion, so we can take $\bar{R} = \bar{R}_1$.

z can be extended to $z' : \bar{R} \to B$ (since $\bar{R}_0$ is injective); define w' to be the composition:

$$\bar{\Lambda} \to \bar{R}_0 \oplus \bar{\Lambda} \to \bar{R} = (\bar{R}_0 \oplus \bar{\Lambda})/\{(\lambda a, -\lambda); \lambda \in I\} \xrightarrow{z'} B .$$

Then $w'/I = w$, therefore $[w] = 0 \in \bar{\pi}(I, B)$. But $f \circ u + w = v/I$, thus $f_*([u]) = [v/I] \in \bar{\pi}(I, B)$.

$[v] \in \bar{\pi}(\Lambda, B)$, $f_* : \bar{\pi}(\Lambda, A) \to \bar{\pi}(\Lambda, B)$ is an isomorphism by hypothesis, so $\exists u_1 : \Lambda \to A$, $w_1 : \bar{\Lambda} \to B$ such that $f \circ u_1 + w_1/\Lambda = v$. In this way $f_*([u_1/I]) = f_*([v/I)] = f_*([u])$ and so $[u - u_1/I] = 0$ (because f_* is isomorphic) i.e. $\exists u_2 : \bar{\Lambda} \to A$, $u_2/I = u - u_1/I$.

Let $u_3 = u_1 + u_2/\Lambda : \Lambda \to A$, $w_2 = w_1 - f \circ u_2 : \bar{\Lambda} \to B$, then $f \circ u_3 + w_2/\Lambda = f \circ (u_1 + u_2/\Lambda) + w_1/\Lambda - f \circ u_2 \Lambda = f \circ u_1 - w_1/\Lambda = v$, $u_3/I = u$ and $w_2/I = w$. Define $x_1 : R \to A$ and $z_1 : \bar{R} \to B$ by $x_1(a_0 + \lambda a) = x(a_0) + u_3(\lambda)$ ($a_0 \in R_0$, $\lambda \in \Lambda$), $z_1([\bar{a}_0, \bar{\lambda}]) = z(\bar{a}_0) + w_2(\bar{\lambda})$ ($\bar{a}_0 \in \bar{R}_0$, $\bar{\lambda} \in \bar{\Lambda}$). Then x_1, z_1 satisfy our requirements. The proof is over.

For any injective module $\bar{A}$, we define the *i-product module associated with* $\bar{A}$ to be the module:

$$\Pi^i \bar{A} = \prod_{\substack{Q \subset \bar{A} \\ Q \text{ is injective}}} Q$$

$\Pi^i \bar{A}$ is naturally an injective module; if $i : \bar{A}_1 \to \bar{A}_2$, then we have a canonical inclusion:

$$\Pi^i i : \Pi^i \bar{A}_1 \to \Pi^i \bar{A}_2 = (\Pi^i \bar{A}_1) \Pi \left(\prod_{\substack{Q \subset \bar{A}_2,\ Q \not\subset \bar{A}_1 \\ Q \text{ is injective}}} Q \right)$$

Moreover, if $i = i_1 \circ i_2$, then $\Pi^i i = (\Pi^i i_1) \circ (\Pi^i i_2)$.

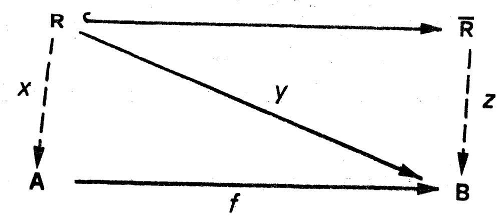
We will show that the weak i-homotopy equivalences coincide with i-homotopy equivalences if the ring Λ satisfies (*) and

(**) $\left\{ \begin{array}{l} \text{for each family of injective submodules } (\bar{R}_l)_{l \in \mathcal{L}} \text{ of a } \Lambda\text{-module} \\ \text{such that } \forall l_1, l_2 \in \mathcal{L},\ \bar{R}_{l_1} \subset \bar{R}_{l_2} \text{ or } \bar{R}_{l_2} \subset \bar{R}_{l_1}, \text{ the module } \bigcup_{l \in \mathcal{L}} \Pi^i \bar{R}_l \\ \text{is injective.} \end{array} \right.$

In virtue of Proposition 2, we see that if Λ is a Noetherian ring, then (**) holds. Particularly, a principal ring satisfies (**).

LEMMA 2. *If Λ verifies (*) and (**), and if $f : A \to B$ is a weak i-homotopy equivalence, then for any module R, $f_* : \bar{\pi}(R, A) \to \bar{\pi}(R, B)$ is surjective.*

Proof. Let $[y] \in \pi(R, B)$, $y : R \to B$, we must find some $x : R \to A$ and some $z : \bar{R} \to B$ such that $f \circ x + z/R = y$:



Let $\mathscr{F} = \{(R_l, \bar{R}_l, r_l, x_l, z_l)$; $R_l, \bar{R}_l, r_l, x_l, z_l$ satisfy i)–iv) below$\}$ $= \{(R_l, \bar{R}_l, r_l, x_l, z_l)$; $l \in \mathscr{M}\}$ ($\mathscr{M}$ is an indexing set for $\mathscr{F}$).

i) R_l is a submodule of R;

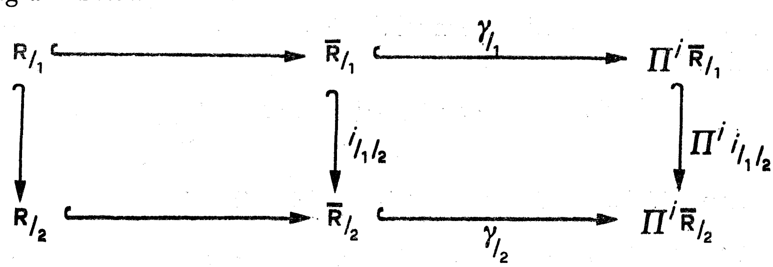
ii) $\bar{R}_l$ is some injective module containing R_l;

iii) $r_l : \bar{R}_l \to \Pi^i \bar{R}_l$ is an inclusion map (not necessarily the canonical one);

iv) $x_l : R_l \to A$, $z_l : \Pi^i \bar{R}_l \to B$ are maps satisfying $f \circ x_l + z_l \circ r_l / R_l = y/R_l$.

Clearly, $\mathscr{F} \neq \Phi$. We define an ordering in $\mathscr{M}$ by
$l_1 \leq l_2$ if and only if: $l_1 = l_2$ or

v) $R_{l_1} \subsetneqq R_{l_2}$, there is an inclusion $i_{l_1 l_2} : \bar{R}_{l_1} \to \bar{R}_{l_2}$ [1] such that the diagram below is commutative:



and that $x_{l_2}/R_{l_1} = x_{l_1}$, $z_{l_2} \circ (\Pi^i i_{l_1 l_2}) = z_{l_1}$.

Let $T(\mathscr{M}) = \{(\mathscr{M}_1, \mu)$; $\mathscr{M}_1 \subset \mathscr{M}$, $\mathscr{M}_1$ is totally ordered, μ satisfies vi) and vii) below$\}$.

vi) μ associates any $(l_1, l_2) \in \{(l_3, l_4) \in \mathscr{M}_1 \times \mathscr{M}_1$; $l_3 < l_4\}$ an inclusion $\mu(l_1, l_2) = i_{l_1 l_2} : \bar{R}_{l_1} \to \bar{R}_{l_2}$ satisfying v);

(1) $i_{l_1 l_2}$ need not be unique.

vii) $\forall l_1, l_2, l_3 \in \mathscr{M}_1, l_1 < l_2 < l_3$, we have $\mu(l_1, l_3) = \mu(l_2, l_3) \circ \mu(l_1, l_2)$. Certainly $T(\mathscr{M}) \neq \Phi$, and $T(\mathscr{M})$ ordered by inclusion (in the obvious sense) is inductive, thus there exists a maximal element $(\mathscr{L}, \mu)$ of $T(\mathscr{M})$. We will denote $i_{l_1 l_2}$ for $\mu(l_1, l_2)$.

Let $R_\infty = \bigcup_{l \in \mathscr{L}} R_l$, R_∞ is a submodule of R. We have:

(A) $\exists l_0 \in \mathscr{L}$ *such that* $R_\infty = R_{l_0}$.

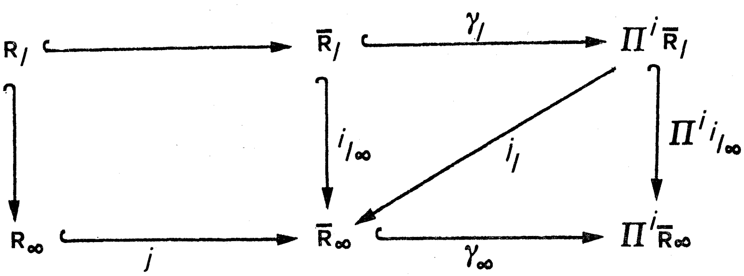
Suppose contrarily that $R_\infty \neq R_l, \forall l \in \mathscr{L}$. Let $\overline{R}_\infty = \bigcup_{l \in \mathscr{L}} \Pi^i \overline{R}_l$ (for any $l_1 < l_2$, $\overline{R}_{l_1}$ is included in $\overline{R}_{l_2}$ by $i_{l_1 l_2} = \mu(l_1, l_2)$, and by vii) we can construct the « union » $\bigcup_{l \in \mathscr{L}} \Pi^i \overline{R}_l$), $\overline{R}_\infty$ is an injective module by (**). Let $i_{l\infty} : \overline{R}_l \to \overline{R}_\infty$ be the composition:

$$\overline{R}_l \xrightarrow{\gamma_l} \Pi^i \overline{R} \xrightarrow{j_l} \overline{R}_\infty = \bigcup_{l' \in \mathscr{L}} \Sigma_i \overline{R}_{l'} .$$

Let $r_\infty : \overline{R}_\infty \to \Pi^i \overline{R}_\infty$ be the map verifying $r_\infty/_{\Pi^i \overline{R}_l} = \Pi^i (i_{l\infty}) = (\Pi^i j_l) \circ (\Pi^i r_l)$:

$$\Pi^i \overline{R}_l \xrightarrow{\Pi^i \gamma_l} \Pi^i (\Pi^i \overline{R}_l) \xrightarrow{\Pi^i j_l} \Pi^i \overline{R}_\infty .$$

Define $j : R_\infty \to \overline{R}_\infty$ by $j/R_l = i_{l\infty}/R_l$. Observe that $i_{l\infty}, r_\infty, j$ are all inclusions, and we have the following commutative diagram for any $l \in \mathscr{L}$:



Define $x_\infty : R_\infty \to A$, $z'_\infty : \overline{R}_\infty \to B$ by $x/R_l = x_l$, $z'_\infty / \Pi^i \overline{R}_l = z_l$ respectively. $\exists z_\infty : \Pi^i R_\infty \to B$, $z_\infty \circ r_\infty = z'_\infty$. From $f \circ x_l + z_l \circ r_l/R_l = y/R_l$, we deduce $f \circ x_\infty + z_\infty \circ r_\infty/R_\infty = y/R_\infty$, we conclude then $(R_\infty, \overline{R}_\infty, r_\infty, x_\infty, z_\infty) \in \in \mathscr{F}$, and so $\exists l'_0 \in \mathscr{M}$ such that $(R_\infty, \overline{R}_\infty, r_\infty, x_\infty, z_\infty) = (R_{l'_0}, R_{l'_0}, r_{l'_0}\, x_{l'_0}, z_{l'_0})$.

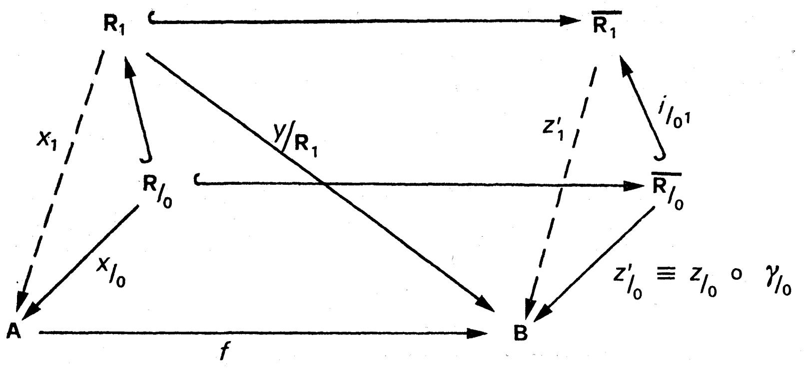
Furthermore $l < l'_0$, $\forall l \in \mathscr{L}$, and if we set $\tilde{\mathscr{L}} = \mathscr{L} \cup \{l'_0\}$ and define $\tilde{\mu}$ by :

$$\tilde{\mu}(l_1, l_2) = \begin{cases} \mu(l_1, l_2) & \text{if } l_1, l_2 \in \mathscr{L}, l_1 < l_2 \\ i_{l_1 \infty} & \text{if } l_1 \in \mathscr{L}, l_2 = l'_0 \end{cases}$$

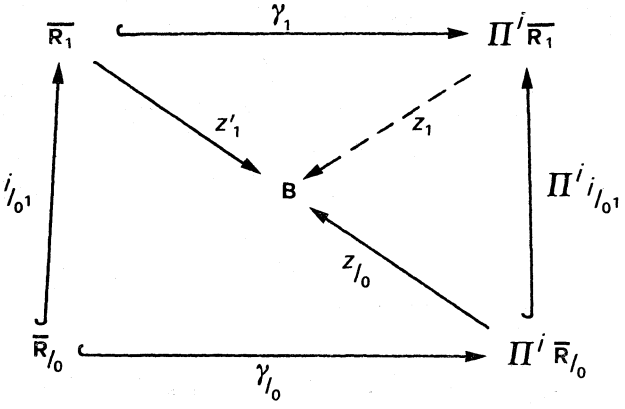
then $(\tilde{\mathscr{L}}, \tilde{\mu}) \in T(\mathscr{M})$, which strictly includes $(\mathscr{L}, \mu)$. But $(\mathscr{L}, \mu)$ is a maximal element of $T(\mathscr{M})$, the contradiction shows that $\exists l_0 \in \mathscr{L}$ such that $R_\infty = R_{l_0}$.

(B) *We claim that* $R_{l_0} = R$.

In fact, if $R_{l_0} \neq R$, choose any $a \in R - R_{l_0}$ and set $R_1 = R_{l_0} + \wedge a$. Using Lemma 1, we may construct $x_1 : R_1 \to A$ and $z'_1 : \bar{R}_1 \to B$ $(\bar{R}_1 \supset \bar{R}_{l_0})$ such that $x_1/R_{l_0} = x_{l_0}$, $z'_1/\bar{R}_{l_0} = z'_{l_0} \equiv z^0_{l_0}\gamma_{l_0}$ and $f \circ x_1 + z'_1/R_1 = y/R_1$:



Now $\bar{R}_1 \simeq \bar{R}_{l_0} \oplus (\bar{R}_1/\bar{R}_{l_0})$, so we can demonstrate the existence of some inclusion $r_1 : \bar{R}_1 \to \Pi^i \bar{R}_1$ which satisfies $r_1 \circ i_{l_01} = (\Pi^i i_{l_01}) \circ r_{l_0}$. In virtue of $z'_1 \circ i_{l_01} = z_{l_0} \circ r_{l_0}$, we get a map $z_1 : \Pi^i \bar{R}_1 \to B$ such that $z_1 \circ r_1 = z'_1$ and $z_1 \circ (\Pi^i i_{l_01}) = z_{l_0}$:



Then $(R_1, \bar{R}_1, r_1, x_1, z_1) \in \mathscr{F}$, let $l'_0 \in \mathscr{M} : (R_1, \bar{R}_1, r_1, x_1, z_1) = (R_{l'_0}, \bar{R}_{l'_0}, r_{l'_0}, x_{l'_0}, z_{l'_0})$, then $l_0 \leq l'_0$, $l_0 \neq l'_0$. As before, we deduce a contradiction to the maximality of $(\mathscr{L}, \mu)$, hence $R_{l_0} = R$.

Finally, since $R_{l_0} = R$, we can take $\bar{R} = \bar{R}_{l_0}$, $x = x_{l_0}$, $z = z_{l_0} \circ r_{l_0}$. It follows that $f \circ x + z/R = y$. This completes the proof.

Now we are ready to prove our main theorem.

THEOREM 1. *Assume that* $\wedge$ *satisfies* (*) *and* (**), $f : A \to B$ *is an i–homotopy equivalence if and only if f is a weak i–homotopy equivalence.*

Proof. We need to prove the « if » part. Let f be a weak i–homotopy equivalence. By Lemma 2, $f_* : \bar{\pi}(B, A) \to \bar{\pi}(B, B)$ is surjective. In particular, there is a map $g : \mathrm{B} \to \mathrm{A}$ such that $f \circ g \simeq_i 1_{\mathrm{B}}$. Then for *any* module R , $f_* \circ g_* = 1_{\bar{\pi}(\mathrm{R},\mathrm{B})}$:

$$\bar{\pi}(\mathrm{R}, \mathrm{B}) \xrightarrow{g_*} \bar{\pi}(\mathrm{R}, \mathrm{A}) \xrightarrow{f_*} \bar{\pi}(\mathrm{R}, \mathrm{B})$$

so g is also a weak i–homotopy equivalence. Using Lemma 2 again, g_* is surjective; by $f_* \circ g_* = 1_{\bar{\pi}(\mathrm{R},\mathrm{B})}$, g_* is an inclusion. Therefore g_* (and hence f_*) is isomorphic. It follows that f is an i–homotopy equivalence (see [4, Theorem 13.12]).

PROPOSITION 3. *If* $\wedge$ *is a principal ring, then* $f : \mathrm{A} \to \mathrm{B}$ *is an i–homotopy equivalence if and only if* $f_* : \bar{\pi}(\wedge, \mathrm{A}) \to \bar{\pi}(\Lambda, \mathrm{B})$ *is isomorphic.*

Under no restrictions on the ring $\wedge$, however we have:

THEOREM 2. *If* A , B *are finitely generated* $\wedge$*–modules, then the weak i–homotopy equivalences between* A *and* B *coincide with the i–homotopy equivalences between* A *and* B.

In fact, Theorem 2 is a consequence of the proposition below (for which we do not give the proof here).

PROPOSITION 4. *If* $f : \mathrm{A} \to \mathrm{B}$ *is a weak i–homotopy equivalence and* R *is a finitely generated* $\wedge$*–module, Then* $f_* : \bar{\pi}(\mathrm{R}, \mathrm{A}) \to \bar{\pi}(\mathrm{R}, \mathrm{B})$ *is surjective.*

Finally, we propose the following problem:

QUESTION. *Which rings may satisfy* (*), (**) *respectively?*

We remark that if $\bar{\wedge}$ is some injective $\wedge$–module containing $\wedge$, then the condition (*) says that the $\wedge$–module $\bar{\wedge}/\mathrm{I}$ is injective for any ideal I of $\wedge$.

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