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Locally compact modules over compact rings

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Locally compact modules over compact rings.* Nota (*) di NICOLA RODINÒ, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Sia A un anello compatto e sia M un A-modulo localmente compatto. Le dimostrazioni note che M è linearmente topologizzato sembrano alquanto involute ed usano risultati profondi della teoria dei gruppi Abeliani localmente compatti nonché il Teorema di Kaplansky che asserisce che A è linearmente topologizzato. In questa Nota, poggiando sul Teorema di Peter–Weyl, viene esposta una dimostrazione semplice e diretta, della quale il Teorema di Kaplansky è corollario.

INTRODUCTION

Let A be a compact ring and let M be a locally compact module. The result that M is linearly topologized appears in [A] and in [S]. In [A] the proof is based on a result of [GS] and on Kaplansky Theorem ([K]), which states that A is linearly topologized. In [S] the short proof is based on the Structure Theorem for locally compact Abelian groups ([HR], Theorem 9.14). In this note we produce an easy and general proof of the considered result. In our proof the role played by Peter-Weyl Theorem is evident and Kaplansky Theorem is obtained as corollary.

In the sequel all rings have a non-zero identity, all modules are unitary and all topologies are Hausdorff. A topological module is *linearly topologized* if the open submodules form an open basis at zero. Let A and B topological rings and let $_{A}M_{B}$ be a topological bimodule. Then M is said *linearly topologized* if the open A-B-sub-bimodules form an open basis at zero. A topological ring A

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is said *left* (*right*) *linearly topologized* if respectively the regular modules ${}_A A$ (A_A) are linearly topologized. A is *linearly topologized* if the regular bimodule ${}_A A_A$ is linearly topologized. Let G be an Abelian topological group. G^* is the *character group* of G. Recall that a character of G is a continuous morphism from G to $T = \mathbf{R}/\mathbf{Z}$. A topological Abelian group is called a *Peter-Weyl group*, by short a *P–W–group*, if the characters separate points in G. A topological module is a *P–W–module* if its additive group is a *P–W–group*. Let M be a left module over a ring A and let W be a subset of M. We put $A \cdot W = \{a x : a \in A, x \in W\}$ and denote by AW the set consisting of all finite sums of the type $\sum_i a_i x_i$, $a_i \in A$, $x_i \in W$. Clearly AW is the submodule of M generated by $A \cdot W$.

1. LEMMA 1. *Let* A *be a compact ring and let* M *be a topological left* A–*module. Then for every neighbourhood* V *of zero in* M, *there is a neighbourhood* W *of* 0 *in* M *such that* $A \cdot W \subseteq V$.

Proof. Since the moltiplication is continuous, for each $a \in A$ there are a neighbourhood U_a of a in A and a neighbourhood W_a of 0 in M such that $U_a \cdot W_a \subseteq V$. Let $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$ and put $W = \bigcap_{i=1}^{n} W_{a_i}$. Clearly $A \cdot W$ is contained in V.

LEMMA 2. *Let* M *be a* P–W–*module. Then* M *is totally disconnected.*

Proof. Fix a small neighbourhood U of zero in T. Let $\xi \in M^*$. For Lemma 1, there is a neighbourhood W of 0 in M such that $\xi(A \cdot W) \subseteq U$. If $x \in W$, $\xi(A x)$ is a subgroup of T and is contained in U. Since U is small, $\xi(A x) = 0$ for each $x \in W$, so that:

$$W \subseteq AW \subseteq \mathrm{Ker}(\xi)$$

and Ker (ξ) is a clopen subgroup of M. By assumption M is a P–W–module. So $\bigcap_{\xi \in M^*} \mathrm{Ker}(\xi) = 0$. Since the connected component of 0 is contained in the intersection of all clopen subsets of M, this proves that M is totally disconnected.

THEOREM 3. *Let* A *be a precompact ring and* M *a locally compact left or right module. Then* M *is linearly topologized and moreover open compact submodules constitute an open basis at zero.*

Proof. We give the proof for a left A–module, the right case being analogous. Suppose first that A is compact. According to Peter-Weyl Theorem ([P]), M is a P–W–module, so that for Lemma 2, M is totally disconnected.

It is known that a totally disconnected locally compact group is linearly topologized ([P]). Let V be any open subgroup of M. For Lemma 1 there is an open neighbourhood W of 0 such that $A \cdot W \subseteq V$. Since V is a subgroup of M, the subgroup AW generated by A . W is contained in V. Now AW is a submodule of M and it is open because W is contained in AW (A has an identity). Suppose now that A is only precompact. Let $\tilde{A}$ be the completion of A. Since M is complete, the multiplication extends to $\tilde{A} \times M$, so M is also a locally compact $\tilde{A}$–module. The $\tilde{A}$–module M is linearly topologized and, since every $\tilde{A}$–submodule of M is also an A–submodule, consequently M is linearly topologized. Finally, let V be an arbitrary compact neighbourhood of zero in M. There exists an open submodule W of M contained in V. Since an open subgroup of M is also closed, W is closed in V and therefore it is compact.

COROLLARY 4. *Let* A *and* B *be precompact rings and let* ${}_AM_B$ *be a locally compact* A–B–*bimodule. Then* M *is linearly topologized.*

Proof. Applying Theorem 3, let ${}_AV$ be on open A–submodule of ${}_AM$. Again for Theorem 3, there is an open B–submodule W_B of M_B such that $W \subseteq V$. The A–submodule generated by W is contained in V and clearly is an open A–B–submodule of ${}_AM_B$.

COROLLARY 5 (Kaplansky Theorem [K]). *Let* A *be a precompact ring. Then the open ideals are an open basis at zero.*

Proof. Let $\tilde{A}$ be the completion of A. Apply Corollary 4 to the compact bimodule ${}_A\tilde{A}_A$. Since $\tilde{A}$ is linearly topologized, also ${}_AA_A$ is linearly topologized.

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