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On multifunctions with convex graph

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Analisi matematica. — *On multifunctions with convex graph.* Nota (*) di BIAGIO RICCERI, (**) presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota viene stabilita una caratterizzazione generale della semicontinuità inferiore delle multifunzioni, a grafico convesso, definite in sottoinsieme non vuoto, aperto e convesso di uno spazio vettoriale topologico e a valori in uno spazio vettoriale topologico localmente convesso. Sono poste in luce, poi, varie conseguenze di tale caratterizzazione.

INTRODUCTION

Multifunctions with convex graph play an important role in various questions of convex analysis (see, for instance, the recent [1] and the bibliography cited there).

The purpose of this Note is to establish a general characterization of the lower semi-continuity of such multifunctions, when the domain space is an open convex subset of a topological vector space and the range space is a locally convex topological vector space. We then derive various consequences. For instance, given a non-empty open convex subset X of $\mathbf{R}^n$ and a non-empty convex subset T of a locally convex topological vector space, we prove that any convex set $S \subseteq X \times T$ is closed in $X \times T$ provided, for each $x \in X$, the set $\{y \in T : (x, y) \in S\}$ is non-empty and closed in T.

1. BASIC DEFINITIONS AND PRELIMINARY RESULTS

Let X, Y be two non-empty sets. A multifunction from X into Y is a function from X into the family of all non-empty subsets of Y. Let F be a multifunction from X into Y (briefly, $F : X \to 2^Y$). For every $A \subseteq X$ and $\Omega \subseteq Y$, we put $F(A) = \bigcup_{x \in A} F(x)$, $F^-(\Omega) = \{x \in X : F(x) \cap \Omega \neq \emptyset\}$, $F^+(\Omega) = \{x \in X : F(x) \subseteq \Omega\}$. If $F(X) = Y$, we say that F is onto Y. The set $\{(x, y) \in X \times Y : y \in F(x)\}$ is called the graph of F and is denoted by gr(F). If F is onto Y, we denote by I_F the inverse multifunction of F, defined by putting $I_F(y) = F^-(y)$ for all $y \in Y$. Of course, we have $I_F(\Omega) = F^-(\Omega)$ for all $\Omega \subseteq Y$. Suppose now that X, Y are two topological spaces. Let $x_0 \in X$. If

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for every open set $\Omega \subseteq Y$, the relation $x_0 \in F^-(\Omega)$ (resp. $x_0 \in F^+(\Omega)$) implies $x_0 \in \operatorname{int}(F^-(\Omega))$ (resp. $x_0 \in \operatorname{int}(F^+(\Omega))$), we say that F is lower (resp. upper) semi-continuous at x_0. F is said to be lower (resp. upper) semi-continuous if it is so at every point of X. We say that F is open (resp. closed), if, for every open (resp. closed) set $A \subseteq X$, the set $F(A)$ is open (resp. closed) in Y. Of course, if F is onto Y, the lower (resp. upper) semi-continuity of F is equivalent to the openness (resp. closure) of I_F. Finally, given $\Phi : X \to 2^Y$, we say that Φ is a multiselection of F if $\Phi(x) \subseteq F(x)$ for all $x \in X$.

Before stating our preliminary results, we point out that the notions of lower and upper semi-continuity, when referred to single-valued real functions, are the classical ones. Moreover, if d is a pseudo-metric on a set Y, for all $y \in Y$, $\Omega \subseteq Y$ $(\Omega \neq \emptyset)$, $\varepsilon > 0$, we put $d(y, \Omega) = \inf\{d(y, z) : z \in \Omega\}$, $B(\Omega, \varepsilon, d) = \{v \in Y : d(v, \Omega) < \varepsilon\}$. For the remainder, our terminology will be that of [2].

We now start by proving the following result.

THEOREM 1.1. *Let* X, Y *be two topological spaces and let* F *be a multifunction from* X *into* Y. *If* Y *is uniformizable, then the following are equivalent:*

(1) F *is lower semi-continuous.*

(2) *There exists a saturated family* $\{d_i\}_{i \in I}$ *of pseudo-metrics on* Y, *generating the topology of* Y, *such that, for every* $i \in I$ *and* $y \in Y$, *the real function* $d_i(y, F(\cdot))$ *is upper semi-continuous.*

Proof. Let us show that (1) ⇒ (2). Since Y is uniformizable, there exists a saturated family $\{d_i\}_{i \in I}$ of pseudo-metrics on Y, generating the topology of Y. Fix $i \in I$, $y \in Y$, $x^* \in X$. Given $\varepsilon > 0$, let $z^* \in F(x^*)$ be such that $d_i(y, z^*) < d_i(y, F(x^*)) + \varepsilon/2$. Since F is lower semi-continuous and $B(z^*, \varepsilon/2, d_i)$ is open in Y, there exists a neighbourhood U of x^* such that $F(x) \cap B(z^*, \varepsilon/2, d_i) \neq \emptyset$ for all $x \in U$. Hence, if $x \in U$, chosen $v_x \in F(x) \cap B(z^*, \varepsilon/2, d_i)$, we have $d_i(y, F(x)) \leq d_i(y, v_x) \leq d_i(y, z^*) + d_i(v_x, z^*) < d_i(F(y, F(x^*)) + \varepsilon$, and so the function $d_i(y, F(\cdot))$ is upper semi-continuous at x^*.

Now, let us show that (2) ⇒ (1). Therefore, let $\{d_i\}_{i \in I}$ be a family of pseudo-metrics as in (2). Let $x_0 \in X$ and let Ω be any open subset of Y such that $F(x_0) \cap \Omega \neq \emptyset$. Choose $y_0 \in F(x_0) \cap \Omega$. Since the family $\{d_i\}_{i \in I}$ is saturated, there exist $i \in I$ and $\varepsilon > 0$ such that $B(y_0, \varepsilon, d_i) \subseteq \Omega$. Since the real function $d_i(y_0, F(\cdot))$ is upper semi-continuous at x_0 and $d_i(y_0, F(x_0)) = 0$, there exists a neighbourhood V of x_0 such that $d_i(y_0, F(x)) < \varepsilon$ for all $x \in V$. Hence, if $x \in V$, we have $F(x) \cap B(y_0, \varepsilon, d_i) \neq \emptyset$, and so, a fortiori, $F(x) \cap \Omega \neq \emptyset$. This completes the proof.

Now, we prove the following result.

THEOREM 1.2. *Let* X, Y *be two topological spaces and let* F *be an upper semi-continuous multifunction from* X *into* Y. *Then, for every* $y \in Y$ *and every* *pseudo-metric* d *on* Y, *generating a weaker topology than that of* Y, *the real function* $d(y, F(\cdot))$ *is lower semi-continuous.*

Proof. Let $y \in Y$ and d be as in the statement. Fix $x^* \in X$ and $\varepsilon > 0$. By hypothesis, the set $B(F(x^*), \varepsilon/2, d)$ is open in Y. Hence, since F is upper semi-continuous at x^*, there exists a neighbourhood U of x^* such that $F(x) \subseteq B(F(x^*), \varepsilon/2, d)$ for all $x \in U$. Therefore, if $x \in U$, for each $z \in F(x)$ there is $v_z \in F(x^*)$ such that $d(z, v_z) < \varepsilon/2$. Thus, we have $d(y, F(x^*)) - \varepsilon/2 < d(y, v_z) - d(z, v_z) \leq d(y, z)$. Since this holds for any $z \in F(x)$, we have $d(y, F(x^*)) - \varepsilon < d(y, F(x))$. This completes the proof.

Remark 1.1. Particular cases of Theorems 1.1 and 1.2 are already known (see, for instance, Theorems 1.2.19 and 1.2.20 of [3]).

The following result will be useful in the sequel.

PROPOSITION 1.1. *Let* X *be a topological space*, (Y, d) *a pseudo-metric space*, $F: X \to 2^Y$. *If, for every* $y \in Y$, *the real function* $d(y, F(\cdot))$ *is lower* (*resp. upper*) *semi-continuous, then the function* $d(\cdot, F(\cdot))$ *is lower* (*resp. upper*) *semi-continuous on* $X \times Y$.

Proof. Let $(x^*, y^*) \in X \times Y$ and $\varepsilon > 0$. Since the function $d(y^*, F(\cdot))$ is lower semi-continuous at x^*, there exists a neighbourhood U of x^* such that $d(y^*, F(x^*)) < d(y^*, F(x)) + \varepsilon/2$ for all $x \in U$. Thus, for all $(x, y) \in U \times B(y^*, \varepsilon/2, d)$ we have $d(y^*, F(x^*)) < d(y^*, F(x)) + \varepsilon/2 \leq \leq d(y, F(x)) + d(y, y^*) + \varepsilon/2 < d(y, F(x)) + \varepsilon$ and so the function $d(\cdot, F(\cdot))$ is lower semi-continuous at (x^*, y^*). The proof in the case of upper semi-continuity is analogous.

Finally, we recall the following proposition (see Theorem 3.15 of [4]).

PROPOSITION 1.2. *Let* X, Y *be two topological spaces and let* $F: X \to 2^Y$ *be a multifunction whose graph is closed in* $X \times F(X)$. *Then, if* $K \subseteq X$ *and* $K_1 \subseteq F(X)$ *are compact, the sets* $F(K)$ *and* $F^-(K_1)$ *are closed, respectively, in* $F(X)$ *and* X.

2. MAIN RESULT AND ITS CONSEQUENCES

Throughout this section, X is a non-empty open convex subset of a topological vector space E, Y is a locally convex topological vector space, F is a multifunction from X into Y with convex graph.

Our main result is the following.

THEOREM 2.1. *The following are equivalent*:

(1) F *is lower semi-continuous.*

(2) F *is lower semi-continuous and the set* $\{(x, y) \in X \times Y : y \in \overline{F(x)}\}$ *is closed in* $X \times Y$.

(3) *There exist a saturated family* $\{p_i\}_{i\in I}$ *of semi-norms on* Y, *generating the topology of* Y, *and a family* $\{\Omega_i\}_{i\in I}$ *of subsets of* Y *such that, for every* $i \in I$, *the set* $p_i(\Omega_i)$ *is bounded and* $\mathrm{int}(F^-(\Omega_i)) \neq \emptyset$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious.

Let us show that (1) $\Rightarrow$ (3). Since Y is locally convex, there exists a saturated family $\{p_i\}_{i\in I}$ of semi-norms on Y generating the topology of Y. Let $x_0 \in X$, $y_0 \in F(x_0)$ and $\varepsilon > 0$. For every $i \in I$, if we put $\Omega_i = B(y_0, \varepsilon, d_i)$, where, of course, d_i denote the pseudo-metric on Y induced by p_i, we have that $p_i(\Omega_i)$ is bounded and that $\mathrm{int}(F^-(\Omega_i)) \neq \emptyset$, since F is lower semi-continuous at x_0.

Finally, let us show that (3) $\Rightarrow$ (2). Therefore, let $\{p_i\}_{i\in I}$ and $\{\Omega_i\}_{i\in I}$ be as in (3). Fix $i \in I$ and $y \in Y$. Let us prove that the real function $d_i(y, F(\cdot))$ is convex (d_i, of course, has the same meaning as above). Indeed, let $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. Given $\varepsilon > 0$, let $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ be such that $d_i(y, y_j) < d_i(y, F(x_j)) + \varepsilon$, $j = 1, 2$. Since the graph of F is convex, we have $\lambda y_1 + (1-\lambda) y_2 \in F(\lambda x_1 + (1-\lambda) x_2)$. Then we have $d_i(y, F(\lambda x_1 + (1-\lambda) x_2)) \leq p_i(y - (\lambda y_1 + (1-\lambda) y_2)) \leq \lambda p_i(y - y_1) + (1-\lambda) p_i(y - y_2) \leq \lambda d_i(y, F(x_1)) + (1-\lambda) d_i(y, F(x_2)) + \varepsilon$. Our claim follows since ε is arbitrary. Now, observe that, for every $x \in \mathrm{int}(F^-(\Omega_i))$, chosen $z_x \in F(x) \cap \Omega_i$, we have $d_i(y, F(x)) \leq p_i(y - z_x) \leq \sup p_i(\Omega_i) + p_i(y)$, that is the function $d_i(y, F(\cdot))$ is bounded on the non-empty open set $\mathrm{int}(F^-(\Omega_i))$. Hence, by a well-known result (see, for instance, Proposition 19.9 of [2]), the function $d_i(y, F(\cdot))$ is continuous. This holds for every $i \in I$ and $y \in Y$, and so, by Theorem 1.1, the multifunction F is lower semi-continuous. Moreover, we have

$$\{(x, y) \in X \times Y : y \in \overline{F(x)}\} = \bigcap_{i\in I} \bigcap_{r>0} \{(x, y) \in X \times Y : d_i(y, F(x)) \leq r\}.$$

By Proposition 1.1, it follows that each function $d_i(\cdot, F(\cdot))$ is continuous on $X \times Y$. Hence, for every $i \in I$ and $r > 0$, the set $\{(x, y) \in X \times Y : d_i(y, F(x)) \leq r\}$ is closed in $X \times Y$, and so also the set $\{(x, y) \in X \times Y : y \in \overline{F(x)}\}$ is closed in $X \times Y$.

Remark 2.1. Observe that, when Y is semi-normable, condition (3) of Theorem 2.1 can be reformulated simply as follows: there exists a bounded set $\Omega \subseteq Y$ such that $\mathrm{int}(F^-(\Omega)) \neq \emptyset$.

A first consequence of Theorem 2.1 is the following

THEOREM 2.2. *Let* F *be lower semi-continuous and let* K *be a compact subset of* X, K_1 *a compact subset of* F(X) *and* Ω *a subset of* Y *containing* F(X). *If, for every* $x \in X$, *the set* F(x) *is closed in* Ω, *then the graph of* F *is closed in* $X \times \Omega$ *and the sets* F(K), $F^-(K_1)$ *are closed, respectively, in* F(X) *and* X.

Proof. By the implication (1) $\Rightarrow$ (2) of Theorem 2.1, the set $\{(x, y) \in X \times \Omega : y \in \overline{F(x)}\}$ is closed in $X \times \Omega$. On the other hand, if, for every

$x \in X$, the set $F(x)$ is closed in Ω, we have $\{(x, y) \in X \times \Omega : y \in \overline{F(x)}\} = \{(x, y) \in X \times \Omega : y \in F(x)\} = \mathrm{gr}(F)$. The last part of our claim follows from Proposition 1.2.

Now, we point out some remarkable particular cases.

THEOREM 2.3. *Let* $\dim(E) < +\infty$. *Then,* F *is lower semi-continuous.*

Proof. This follows at once from Theorem 2.1 and from Corollary 19.10 of [2].

THEOREM 2.4. *Let* Y *be semi-normable. If* F *admits a multiselection* Φ *upper semi-continuous at some point* $x_0 \in X$ *such that* $\Phi(x_0)$ *is bounded, then* F *is lower semi-continuous.*

Proof. Since $\Phi(x_0)$ is bounded and Y is semi-normable, there is a bounded open set $\Omega \subseteq Y$ such that $\Phi(x_0) \subseteq \Omega$. By the upper semi-continuity of Φ at x_0, we have $\mathrm{int}(\Phi^+(\Omega)) \neq \varnothing$, and so, a fortiori, $\mathrm{int}(F^-(\Omega) \neq \varnothing$, since $\Phi^+(\Omega) \subseteq F^-(\Omega)$. Now, our claim follows from the implication (3) $\Rightarrow$ (1) of Theorem 2.1, taking into account Remark 2.1.

We want observe that, in Theorem 2.4, the boundedness assumption on $\Phi(x_0)$ is essential. In fact, we have the following characterization.

THEOREM 2.5. *The following are equivalent:*

(1) *Any lower semi-continuous convex real function on* X *is continuous.*

(2) *For every locally convex topological vector space* Σ, *any upper semi-continuous multifunction from* X *into* Σ, *with convex graph, is lower semi-continuous.*

Proof. Let us prove that (1) $\Rightarrow$ (2). Let $\{p_i\}_{i\,I}$ be a saturated family of semi-norms on Σ generating the topology of Σ. Let $H : X \to 2^{\Sigma}$ be upper semi-continuous and with convex graph. By the proof of Theorem 2.1 and by Theorem 1.2, for every $i \in I$ and $\sigma \in \Sigma$, the real function $d_i(\sigma, H(\cdot))$ is convex and lower semi-continuous and so, by (1), continuous. Hence, by Theorem 1.1, H is lower semi-continuous.

Now, let us prove that (2) $\Rightarrow$ (1). Assume the contrary. Therefore, let f be a lower semi-continuous convex real function on X which is not continuous. Let $\Sigma = \mathbf{R}$ and $H(x) = [f(x), +\infty[$ for all $x \in X$. Since f is convex, the graph of H is convex. The lower semi-continuity of f implies the upper semi-continuity of H. However, since f is not continuous, one can check that H is not lower semi-continuous. This completes the proof.

Now, we want to stress how the above results can be useful to recognize if a given convex set $S \subseteq X \times Y$ is closed. For the sake of brevity, we limit ourselves to an application of Theorems 2.2 and 2.3.

THEOREM 2.6. *Let* $\dim(E) < +\infty$ *and let* S *be a convex subset of* $X \times Y$ *and* Ω *a subset of* Y *containing the projection of* S *on* Y. *If, for every* $x \in X$, *the set* $\{y \in Y : (x, y) \in S\}$ *is non-empty and closed in* Ω, *then* S *is closed in* $X \times \Omega$.

Proof. For each $x \in X$, put $H(x) = \{y \in Y : (x, y) \in S\}$. Since $\mathrm{gr}(H) = S$, by Theorem 2.3, the multifunction H is lower semi-continuous. Now, the conclusion follows directly from Theorem 2.2.

Among the consequences of Theorem 2.6, we point out the following.

THEOREM 2.7. *Let* $\dim(E) < +\infty$ *and let* Ω *be a non-empty convex subset of* Y. *Let* f *be any quasi-convex real function on* $X \times \Omega$ *such that, for every* $x \in X$, *the function* $f(x, \cdot)$ *is lower semi-continuous and unbounded below. Then,* f *is lower semi-continuous on* $X \times \Omega$.

Proof. Let $r \in \mathbf{R}$. Since f is quasi-convex on $X \times \Omega$, the set $f^{-1}(]-\infty, r])$ is convex. The other assumptions made on f imply that, for every $x \in X$, the set $\{y \in \Omega : (x, y) \in f^{-1}(]-\infty, r])\}$ is non-empty and closed in Ω. Hence, by Theorem 2.6, $f^{-1}(]-\infty, r])$ is closed in $X \times \Omega$. This completes the proof.

From now on, T is a non-empty convex subset of Y and G is a multifunction from T onto X, with convex graph. If we apply our preceding theorems by taking $F = I_G$, we obtain corresponding openness results for G.

THEOREM 2.8. *The following are equivalent:*

(1) G *is open.*

(2) G *is open and the set* $\{(x, y) \in X \times Y : y \in \overline{G^-(x)}\}$ *is closed in* $X \times Y$.

(3) *There exist a saturated family* $\{p_i\}_{i \in I}$ *of semi-norms on* Y, *generating the topology of* Y, *and a family* $\{\Omega_i\}_{i \in I}$ *of subsets of* T *such that, for every* $i \in I$, *the set* $p_i(\Omega_i)$ *is bounded and* $\mathrm{int}(G(\Omega_i)) \neq \varnothing$.

Proof. If we take $F = I_G$, of course, $\mathrm{gr}(F)$ is convex and conditions (1), (2), (3) are equivalent to the homonymous ones of Theorem 2.1, and so they are pairwise equivalent.

The following three theorems are consequences, respectively, of Theorems 2.2, 2.3 and 2.4.

THEOREM 2.9. *Let* G *be open and let* K *be a compact subset of* X, K_1 *a compact subset of* T *and* Ω *a subset of* Y *containing* T. *If, for every* $x \in X$, *the set* $G^-(x)$ *is closed in* Ω, *then the graph of* G *is closed in* $\Omega \times X$ *and the sets* $G^-(K)$, $G(K_1)$ *are closed, respectively, in* T *and* X.

THEOREM 2.10. *Let* $\dim(E) < +\infty$. *Then,* G *is open.*

THEOREM 2.11. *The following are equivalent:*

(1) *Any lower semi-continuous convex real function on* X *is continuous.*

(2) *For every locally convex topological vector space* Σ *and every non-empty convex set* $V \subseteq \Sigma$, *any closed multifunction from* V *onto* X, *with convex graph, is open.*

The final result we want to present is the following.

THEOREM 2.12. *Let* E *be Hausdorff and* T *be* σ*-compact. Moreover, suppose that* G *is upper semi-continuous and that* G (y) *is compact for all* $y \in$ T. *Then, for any vector topology* τ *on* E, *stronger than the original one, such that* (E, τ) *is a Baire space,* G *is open with respect to* τ.

Proof. Let $\{T_n\}$ be a sequence of compact subsets of T such that $T = \bigcup_{n=1}^{\infty} T_n$. Of course, we have $X = \bigcup_{n=1}^{\infty} G(T_n)$. By Theorem 3 on p. 116 of [5], each $G(T_n)$ is compact and so closed in X, since X is Hausdorff. If τ is any topology on E as in the statement, then there exists $\bar{n} \in \mathbf{N}$ such that the τ-interior of $G(T_{\bar{n}})$ is non-empty. Now, if $\{p_i\}_{i \in I}$ is any saturated family of semi-norms on Y generating the topology of Y, then, for every $i \in I$, $p_i(T_{\bar{n}})$ is bounded. Hence, by the implication (3) $\Rightarrow$ (1) of Theorem 2.8, G is open with respect to τ.

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