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Some properties of integral curves in a neighbourhood of planar singular points

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Equazioni differenziali. — *Some properties of integral curves in a neighbourhood of planar singular points.* Nota (*) di YU SHU–XIANG e JIN CHENGFU, presentata dal Corrisp. R. CONTI.

RIASSUNTO. — Si studia l'andamento delle traiettorie di un sistema dinamico piano rappresentato dalle equazioni (1) del testo, nell'intorno di un punto singolare isolato.

I. INTRODUCTION

Consider the differential system defined in the plane

$$\frac{dx}{dt} = P(x, y) \qquad (1)$$
$$\frac{dy}{dt} = Q(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are continuous functions with continuous first partial derivatives. We suppose $P(0, 0) = Q(0, 0) = 0$ and there is a constant $R > 0$ such that

$$F(x, y) = P^2(x, y) + Q^2(x, y) > 0 \quad \text{when} \quad 0 < x^2 + y^2 < R^2. \qquad (2)$$

In the study of the behaviour of integral curves in the neighbourhood of a non-elementary singular point, it is important to know the number of trajectories tending to this point along a given exceptional direction. It is reduced to studying the decision problems for Frommer's normal sectors. A considerable number of papers have been written in connection with these problems (see [1, Ch. V]). In the present paper, we give some new results based on some distinct ideas.

II. THE MAIN RESULTS

In addition, we impose the following hypothesis.

(H). There exists a constant $\alpha_1 > 0$ such that any curve of the family

$$\mathcal{T}_\alpha = \{F(x, y) = \alpha \mid (x, y) \in (2),\ 0 < \alpha < \alpha_1\} \qquad (3)$$

is a closed Jordan curve, and $\mathcal{T}_{\alpha_i}$ is contained in the domain bounded by $\mathcal{T}_{\alpha_j}$ when $0 < \alpha_i < \alpha_j < \alpha_1$.

Consider now the system (1). With every point $M = (x, y)$ of the plane we associate the vector $V(M) = (P, Q)$. Let K be a closed Jordan curve not

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passing through any singular point. Take the counter clockwise sense along K as positive sense; assign a fixed direction β in the plane, say, the positive x-axis; take a fixed point A on K; take any one of the infinitely many values of the angle between the direction β and the vector V (A) and denote its value by ψ. If M traverses K once in the positive sense beginning at A, ψ varies continuously, and since the final position of M coincides with its initial position, the final value of ψ will differ from its initial value by $2\pi j_k$ where j_k is an integer. j_k is called the Kronecker index of K with respect to the system (1). Instead of considering a closed Jordan curve, we consider an open Jordan arc. By extending the definition of index, we can introduce the notion of variation of the vector V along an arc $L = \widehat{AB}$ of the curve. The variation of V along $\widehat{AB}$ is denoted by W_{AB} (see [1, p. 189]). Clearly, W_{AB} is the variation of V along $\widehat{AB}$ from A to B.

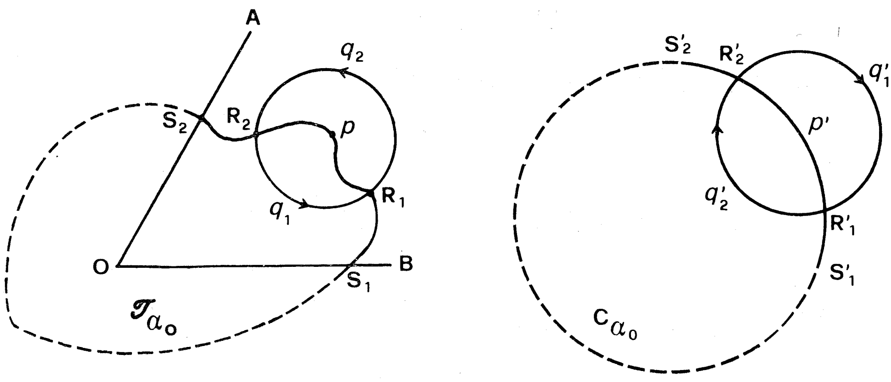
The functions $P(x, y)$ and $Q(x, y)$ define a mapping

$$\phi : \quad u = P(x, y) \quad , \quad v = Q(x, y) . \tag{4}$$

Denote the Jacobian $\dfrac{\partial (P, Q)}{\partial (x, y)}$ by $\Delta(x, y)$. Then we have

LEMMA 1. *Suppose that the system* (1) *satisfies the hypothesis* (H). *Let* AOB *be a sectorial region in* (2). *Let* $\widehat{S_1S_2}$ *be a segmental arc of* $\mathcal{T}_{\alpha_0}$ $(0 < \alpha_0 < \alpha_1)$ *which lies in* AOB, *where* $S_1 \in OB$ *and* $S_2 \in OA$, *and such that the sense moving from* S_1 *to* S_2 *along* $\widehat{S_1S_2}$ *coincides with the positive sense of* $\mathcal{T}_{\alpha_0}$. *If the variation* $WS_1S_2 > 0$ (< 0) *then there must be a point* $E \in \widehat{S_1S_2}$ *such that* $\Delta(E) \geq 0$ (≤ 0).

Proof. The proof proceeds by reduction to absurdity. Suppose $\Delta(x, y) < 0$ at each point on the arc $\widehat{S_1S_2}$ Then, ϕ maps $\mathcal{T}_{\alpha_0}$ onto the circumference C_{α_0} in the uv-plane; $\widehat{S_1S_2}$ is mapped onto the segmental arc $\widehat{S_1'S_2'}$ of C_{α_0}, i. e., $\widehat{S_1'S_2'}$ is the image of homeomorphism of $\widehat{S_1S_2}$. From the property of local homeomorphism it follows that there are no double points on $\widehat{S_1'S_2'}$ Thus, by the condition $W_{S_1S_2} > 0$ it follows that the sense moving from S_1' to S_2' along $\widehat{S_1'S_2'}$ coincides with the positive sense of C_{α_0} (i.e., counter clockwise sense)



a) Fig. 1. *b)*

The segmental arcs $\widehat{S_1S_2}$ and $\widehat{S_1'S_2'}$ are shown in fig. 1a and fig. 1b respectively.

Choose an arbitrary point $p \in \widehat{S_1S_2}$, let $\phi(p) = p'$ ($\in \widehat{S_1'S_2'}$). By virtue of a well-known fact (see [2, p. 586]) and $\Delta(p) < 0$ it follows that some neighbourhood of p in (2) is homeomorphically mapped onto a neighbourhood of p' by ϕ, and the mapping degree of ϕ in p' is equal to -1. Further, from the properties of mapping degree (see [2, pp. 568-574 and pp. 73-74]) it follows that there are a neighbourhood $U(p)$ of p and a neighbourhood $U(p')$ of p', each of their boundaries $\partial U(p)$ and $\partial U(p')$ is a simple closed curve, and, ϕ homeomorphically maps $U(p)$ and $\partial U(p)$ onto $U(p')$ and $\partial U(p')$ respectively and such that when M traverses $\partial U(P)$ once in the positive sense, the corresponding point $\phi(M)$ traverses $\partial U(p')$ once in the negative sense (i.e., clockwise sense). Denote $\partial U(p) \cap \widehat{S_1S_2} = \{R_1, R_2\}$ and $\partial U(p') \cap \widehat{S_1'S_2'} = \{R_1', R_2'\}$. Clearly, if the sense moving from R_1 to R_2 along $\widehat{S_1S_2}$ coincides with the positive sense of $\mathscr{T}_{\alpha_0}$, then the sense moving from R_1' to R_2' along $\widehat{S_1'S_2'}$ coincides with the positive sense of C_{α_0} provided that $U(p)$ is small enough (see fig. 1*a*, 1*b*). Since $\partial U(p)$ is homeomorphic to $\partial U(p')$, thus, the external half neighbourhood enclosed by curvilinear figure $R_1\, p\, R_2\, q_2\, R_1$ in xy-plane must be homeomorphic to the internal half neighbourhood enclosed by curvilinear figure $R_1'\, p'\, R_2'\, q_2'\, R_1'$ in uv-plane. But this is impossible, because the condition (H) implies that any point M_0 of the external half neighbourhood lies on the curve $\mathscr{T}_\alpha$ corresponding to $\alpha > \alpha_0$, hence the point $\phi(M_0) = M_0'$ must lie the exterior of the circle C_{α_0} in uv-plane (and therefore it cannot belong to the internal half neighbourhood). So Lemma 1 is proved.

THEOREM 1. *Suppose that the system* (1) *satisfies the hypothesis* (H). *Suppose that an exceptional direction of the singular point* O *is contained in a normal sector* D *of a certain type and suppose* $\Delta(x, y) < 0$ *in* D. *The following conclusions are then valid:*

(i) D *can not be a normal sector of the first type* (fig. 2).

(ii) *Is* D *is a normal sector of the third type, then in* D *there are no trajectories of* (1) *tending to* O *along this exceptional direction* (fig. 3).

Proof. (i) If D is a normal sector of the first type (fig. 2) then the two sides OB_1, OA_1 of the normal sector are both crossed outward (or inward) by trajectories. Consider a closed Jordan curve $\mathscr{T}_{\alpha_2}$ of the family (3) where $0 < \alpha_2 < \alpha_1$. $\widehat{S_3S_4}$ denotes the segmental arc of $\mathscr{T}_{\alpha_2}$ which lies in D and such that the sense moving from S_3 to S_4 along $\widehat{S_3S_4}$ coincides with the positive sense

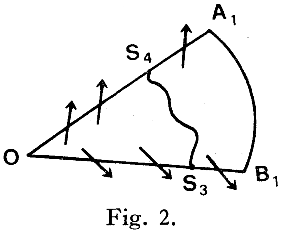


Fig. 2.

of $\mathscr{T}_{\alpha_z}$. It is easy to see that when a point M moves from S_3 to S_4 along $\widehat{S_3S_4}$, the algebraic sum θ of the rotated angle of the vector $V = (P, Q)$ is not less then $< B_1OA_1$ (note that, by definition, the vector $V = (P, Q)$ is not orthogonal to the radius vector OZ at any Z of D. And, in the general case, D can be sufficiently small such that it contains only one exceptional direction). Thus the variation $W_{S_3S_4} > 0$. By applying Lemma 1 it follows that there must be a point $E \in \widehat{S_3S_4}$ such that $\Delta(E) \geq 0$. But this is contradictory to the conditions of Theorem 1. Hence conclusion (i) is proved.

To prove (ii), we suppose that D is a normal sector of the third type and suppose that in D there exists at least one trajectory of (1) which tends to O (and therefore there are an infinite number of trajectories of (1) which tend to O). (see fig. 3).

Let the integral curve OM_3 be a boundary of the parabolic sector adjacent to the singular point O. For a point M_r lying on the integral curve OM_3, the angle $\delta(r)$ between the direction of the vector $V(M_r)$ and the direction of the vector $\overrightarrow{OD_3}$ (it is just the exceptional direction in D) will be sufficiently small provided the radius r is small enough.

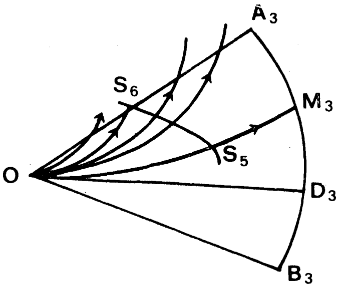


Fig. 3.

Consider now a curve $\mathscr{T}_{\alpha_3}$ of (3) where α_3 is sufficiently small, and consider an its segmental arc $\widehat{S_5S_6}$ which is the intersection of $\mathscr{T}_{\alpha_3}$ and the region bounded by the integral curve OM_3, a side OA_3 of D and the curve arc $\widehat{A_3M_3}$ When M moves from S_5 to S_6 along $\widehat{S_5S_6}$ in the positive sense of $\mathscr{T}_{\alpha_3}$, the algebraic sum θ of the rotated angle of the vector $V = (P, Q)$ is not less than $\angle D_3OA_3 - \delta(\gamma)$. Since α_3 is small enough (hence r is small), $\delta(r)$ is also small, thus $\theta \geq \angle D_3OA_3 - \delta(\gamma) > 0$. Therefore by applying Lemma 1 it follows that there must be a point $E \in \widehat{S_5S_6}$ such that $\Delta(E) \geq 0$. Thus we reach a contradiction with the assumption that $\Delta < 0$ in D and the conclusion (ii) is also proved. Hence Theorem 1 is completely proved.

References

[1] G. Sansone and R. Conti (1964) – *Non-linear Differential Equations*, Pergamon Press Inc. (English).

[2] P.S. Alexandroff (1947) – *Combinatorial Topology* (Russian).