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Ravi P. Agarwal

On the Right Focal Point Boundary Value Problems for Linear Ordinary Differential Equations

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Equazioni differenziali ordinarie. — *On the Right Focal Point Boundary Value Problems for Linear Ordinary Differential Equations.* Nota di RAVI P. AGARWAL, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Scopo della presente Nota è quello di fornire una maggiorazione della lunghezza *b-a* dell'intervallo $[a, b]$ sul quale il problema (1) (2) (3) ammette soltanto la soluzione nulla.

The purpose of this paper is to provide an upper estimate on the length of the interval $(b - a)$ so that the only solution of the linear boundary value problem

$$x^{(n)} + \sum_{i=0}^{n-1} p_i(t)\, x^{(i)} = 0 \qquad (n \geq 2) \tag{1}$$

$$x^{(i)}(a) = 0\,, \quad 0 \leq i \leq k-1 \quad (1 \leq k \leq n-1 \text{ and fixed}) \tag{2}$$

$$x^{(i)}(b) = 0\,, \quad k \leq i \leq n-1 \tag{3}$$

where $p_i \in C[a, b]$, $0 \leq i \leq n-1$ is the trivial solution.

Right focal point (the nomenclature comes from Polynomial interpolation) boundary value problems has been a subject of recent study [1-10] in which necessary and sufficient conditions for the existence and uniqueness of the solutions have been discussed. In this paper we shall prove

THEOREM

$$\text{Let } \sup_{t \in [a,b]} |p_i(t)| \leq M_i\,, \quad 0 \leq i \leq n-1 \quad \text{and} \tag{4}$$

$$\sum_{i=0}^{n-1} M_i\, C_{n,i}^{k}\, (b-a)^{n-i} \leq 1$$

where

$$C_{n,i}^{k} = \frac{1}{(n-i)!} \left| \sum_{j=0}^{k-i-1} \binom{n-i}{j} (-1)^{n-i-j} \right|, \; 0 \leq i \leq k-1 \tag{5}$$

$$= \frac{1}{(n-i)!}\,, \quad k \leq i \leq n-1\,. \tag{6}$$

(*) Nella seduta del 14 dicembre 1985.

Then, the boundary value problem (1)–(3) has only the trivial solution. For this we need the following

LEMMA 1. *The Green's function* $g_k(t, s)$ *of* $x^{(n)} = 0$, (2), (3) *is given by*

$$g_k(t, s) = \frac{1}{(n-1)!} \begin{cases} \sum_{i=0}^{k-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & s \leq t \\ -\sum_{i=k}^{n-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & s \geq t \end{cases} \tag{7}$$

and on $[a, b] \times [a, b]$ *the following inequalities hold*

$$(-1)^{n-k} g_k^{(i)}(t, s) \geq 0, \quad 0 \leq i \leq k-1 \tag{8}$$

$$(-1)^{n-i} g_k^{(i)}(t, s) \geq 0, \quad k \leq i \leq n-1 \tag{9}$$

where $g_k^{(i)}(t, s)$ *denotes the i-th derivative* $\frac{\partial^i}{\partial t^i} g_k(t, s)$.

Proof. It is easy to verify that the function

$$x(t) = \frac{1}{(n-1)!} \left[\int_a^t (t-s)^{n-1} f(s)\, ds - \int_a^b \sum_{i=k}^{n-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1} f(s)\, ds \right]$$

is a solution of the differential equation $x^{(n)} = f(t)$ and $x^{(i)}(a) = 0$, $0 \leq i \leq k-1$. Further, for $0 \leq j \leq n-k-1$ we have

$$x^{(k+j)}(b) = \frac{1}{(n-k-j-1)!} \int_a^b (b-s)^{n-k-j-1} f(s)\, ds -$$

$$- \int_a^b \sum_{i=0}^{n-k-j-1} \frac{(b-a)^i (a-s)^{n-k-j-1-i}}{(n-k-j-1-i)!\, i!} f(s)\, ds$$

$$= \frac{1}{(n-k-j-1)!} \left[\int_a^b (b-s)^{n-k-j-1} f(s)\, ds - \right.$$

$$\left. - \int_a^b ((a-s) + (b-a))^{n-k-j-1} f(s)\, ds \right] =$$

$$= 0 .$$

This function $x(t)$ can also be written as

$$x(t) = \int_a^b g_k(t, s) f(s) \, ds$$

now follows from the equality

$$(t - s)^{n-1} - \sum_{i=k}^{n-1} \binom{n-1}{i} (t - a)^i (a - s)^{n-i-1} = \tag{10}$$

$$= \sum_{i=0}^{k-1} \binom{n-1}{i} (t - a)^i (a - s)^{n-i-1} .$$

The proof of (8) is given in [9] and (9) follows from the explicit representation

$$g_k^{(i)}(t, s) = \frac{1}{(n - i - 1)!} \begin{cases} 0 & , s \leq t \\ -(t - s)^{n-i-1}, & s \geq t \end{cases} \tag{11}$$

obtained by differentiating i-times (7) and (10).

LEMMA 2. *Let* $x \in C^{(n)}[a, b]$, *satisfying* (2), (3). *Then, the following inequalities hold*

$$| x^{(i)}(t) | \leq C_{n,i}^k (b - a)^{n-i} \max_{a \leq t \leq b} | x^{(n)}(t) | , \quad 0 \leq i \leq n - 1 . \tag{12}$$

Proof. Any such function can be written as

$$x(t) = \int_a^b g_k(t, s) \, x^{(n)}(s) \, ds$$

and hence

$$| x^{(i)}(t) | \leq \left(\max_{a \leq t \leq b} \int_a^b | g_k^{(i)}(t, s) | \, ds \right) \max_{a \leq t \leq b} | x^{(n)}(t) | .$$

Thus, it suffices to show that

$$\max_{a \leq t \leq b} \int_a^b | g_k^{(i)}(t, s) | \, ds \leq C_{n,i}^k (b - a)^{n-i} .$$

From (7) and (8) for $0 \leq i \leq k-1$, we have

$$\max_{a\leq t\leq b} \int_a^b | g_k^{(i)}(t,s) | \, ds = \max_{a\leq t\leq b} \frac{1}{(n-1)!} \Bigg| -\sum_{j=i}^{k-1} \binom{n-1}{j} \frac{j!}{(j-i)!} (t-a)^{j-i} \frac{(a-t)^{n-j}}{n-j}$$

$$+ \sum_{j=k}^{n-1} \binom{n-1}{j} \frac{j!}{(j-i)!} (t-a)^{j-i} \frac{(a-b)^{n-j} - (a-t)^{n-j}}{n-j} \Bigg|$$

$$= \max_{a\leq t\leq b} \Bigg| -\sum_{j=i}^{n-1} \frac{(-1)^{n-j}}{(n-j)!\,(j-i)!} (t-a)^{n-i}$$

$$+ \sum_{j=k}^{n-1} \frac{(-1)^{n-j}}{(n-j)!\,(j-i)!} (b-a)^{n-j} (t-a)^{j-i} \Bigg|$$

$$= \frac{1}{(n-i)!} \max_{a\leq t\leq b} \Big| (t-a)^{n-i} + \sum_{j=k-i}^{n-i-1} \binom{n-i}{j} (a-b)^{n-i-j} (t-a)^j \Big|$$

$$= \frac{1}{(n-i)!} \max_{a\leq t\leq b} \Big| \sum_{j=k-i}^{n-i} \binom{n-i}{j} (a-b)^{n-i-j} (t-a)^j \Big|$$

$$= \frac{1}{(n-i)!} \Big| \sum_{j=k-i}^{n-i} \binom{n-i}{j} (-1)^{n-i-j} \Big| (b-a)^{n-i}$$

$$= C_{n,i}^k (b-a)^{n-i} .$$

Similarly, from (9) and (11) for $k \leq i \leq n-1$, we have

$$\max_{a\leq t\leq b} \int_a^b | g_k^{(i)}(t,s) | \, ds = \max_{a\leq t\leq b} \frac{1}{(n-i-1)!} \int_t^b (s-t)^{n-i-1} \, ds$$

$$= \frac{1}{(n-i)!} (b-a)^{n-i} .$$

REMARK. In (12) the constants $C_{n,i}^k$, $0 \leq i \leq n-1$ are the best possible as they are exact for the function

$$x(t) = \frac{1}{n!} \sum_{i=k}^{n} \binom{n}{i} (a-b)^{n-i} (t-a)^i$$

and only for this function up to a constant factor.

PROOF OF THE THEOREM. Suppose on the contrary that (1) – (3) has a non-trivial solution $x(t)$. Then, $M_n = \max_{a \le t \le b} |x^{(n)}(t)| \neq 0$, since otherwise $x(t)$ would coincide with a polynomial of degree $m < n$ on $[a, b]$ and $x^{(m)}(t)$ would not vanish on $[a, b]$ which contradicts the assumption that $x^{(m)}(a) = 0$ (if $0 \le m \le k - 1$) or $x^{(m)}(b) = 0$ (if $k \le m \le n - 1$). Thus, if $M_n = |x^{(n)}(t_1)|$ from the differential equation (1), we have

$$M_n = \left| \sum_{i=0}^{n-1} p_i(t_1)\, x^{(i)}(t_1) \right| \le \sum_{i=0}^{n-1} M_i\, |x^{(i)}(t_1)| . \tag{13}$$

Using Lemma 2 in the above inequality, we get

$$M_n \le \sum_{i=0}^{n-1} M_i\, C_{n,i}^k\, (b - a)^{n-i}\, M_n$$

and hence

$$\sum_{i=0}^{n-1} M_i\, C_{n,i}^k\, (b - a)^{n-i} \ge 1 . \tag{14}$$

To exclude the possibility of equality in (14), we note that at least one of the numbers M_i, $0 \le i \le n - 1$ is different from zero, otherwise again $x(t)$ would be a polynomial of degree less than n and cannot satisfy (2) and (3). Thus, if in (14) equality holds then equality must hold in (12) for at least one value of i. From the remark, this is possible only if $x(t)$ is a polynomial of degree n. Thus, equality in (13) holds for any point t_1 in $[a, b]$ however, $|x^{(i)}(t_1)|$ is not constant on $[a, b]$ for any $0 \le i \le n - 1$ ensures the strict inequality in (14). This completes the proof of our theorem.

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