Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A bound for the solutions of a basic elliptic system with non-linearity $q \geq 2$

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **80** (1986), n.3, p. 81–88.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1986_8_80_3_81_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1986.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'8 marzo 1986

Presiede il Presidente della Classe EDOARDO AMALDI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *A bound for the solutions of a basic elliptic system with non-linearity* $q \geq 2$. Nota di SERGIO CAMPANATO, presentata (*) dal Socio G. FICHERA.

RIASSUNTO. — In questa Nota si dimostra un risultato enunciato nel § 5 della pubblicazione [4].

Per le soluzioni di un sistema ellittico base, con non-linearità $q \geq 2$, vale un principio di massimo analogo a quello dimostrato in [3] nel caso di non-linearità $q = 2$.

1. Let $B(R) = \{x : \|x\| < R\}$ be an open ball in R^n, $n \geq 2$, $N > 1$ an integer, v a vector $B(R) \to R^N$ and $Dv = (D_1 v, \ldots, D_n v)$.

We shall denote by $p = (p^1, \ldots, p^n)$, with $p^i \in R^N$, a generic vector of R^{nN}.

Let q be a real number, with $2 \leq q \leq n$.

For every $p \in R^{nN}$ let us set

$$V(p) = (1 + \|p\|^2)^{1/2} \quad , \quad W(p) = V^{\frac{q-2}{2}}(p)\, p .$$

Consider the second order basic system

$$\sum_i D_i a^i(Dv) = 0 \quad \text{in} \quad B(R) \tag{1}$$

where the vectors $a^i(p) \in R^N$ are of class $C^1(R^{Nn})$.

(*) Nella seduta dell'8 marzo 1986.

By setting

$$A_{ij}(p) = \left\{ \frac{\partial a_h^i(p)}{\partial p_k^j} \right\}_{hk=1,\dots,N}$$

the matrices A_{ij} satisfy the strong ellipticity conditions

$$\left\{ \sum_{ij} \| A_{ij}(p) \|^2 \right\}^{1/2} \leq M V^{q-2}(p), \qquad \forall p \in R^{nN} \tag{2}$$

$$\sum_{ij} (A_{ij}(p)\, \xi^j \mid \xi^i) \geq \nu V^{q-2}(p) \| \xi \|^2, \qquad \forall p, \xi \in R^{nN}. \tag{3}$$

We may assume, without any loss of generality, that $a^i(0) = 0$ so that, because of (2).

$$\| a^i(p) \| \leq M V^{q-2}(p) \| p \| \tag{4}$$

It is well-known that, if $u \in H^{1,q}(B(R))$, the Dirichlet problem

$$\begin{aligned} & v - u \in H_0^{1,q}(B(R)) \\ & \sum_i D_i a^i(Dv) = 0 \qquad \text{in} \quad B(R) \end{aligned} \tag{5}$$

has a unique solution v and the estimate

$$\int_{B(R)} \| W(Dv) \|^2 \, dx \leq c \int_{B(R)} \| W(Du) \|^2 \, dx \tag{6}$$

holds (see the (1.11) in [5]).

Recall that a vector w belongs to the Morrey space $L^{q,\mu}(B(R))$, with $0 \leq \mu \leq n$, if

$$\| w \|^q_{L^{q,\mu}(B(R))} = \sup \sigma^{-\mu} \int_{B(x^0,\sigma) \cap B(R)} \| w \|^q \, dx < +\infty \tag{7}$$

where the supremum is taken over all balls $B(x^0, \sigma)$ with $x^0 \in B(R)$ and $0 < \sigma \leq 2R$.

In this section, we shall prove the following regularity result

THEOREM 1. *If* $v \in H^{1,q}(B(R))$ *is the solution of the Dirichlet problem* (5) *and*

$$2 \leq n \leq q + 2 \tag{8}$$

(9) $$u \in L^\infty(B(R))$$

(10) $$Du \in L^{q,n-q}(B(R))$$

then $v \in L^\infty(B(R))$ *and*

(11) $$\sup_{B(R)} \| v(x) \|^q \leq c \{ \| W(Du) \|^2_{L^{2,n-q}(B(R))} + \sup_{B(R)} \| u(x) \|^q \} .$$

Note that an analogous result is already proved in [7] for the basic linear systems, and in [3] for the basic systems with non-linearity $q = 2$.

Finally, our result is claimed in the § 5 of [4], without any proof.

Proof of the Theorem 1. For $x^0 \in B(R)$, we shall denote by y^0 a point on $\partial B(R)$ such that

$$\| x^0 - y^0 \| = d = \text{dist.}(x^0, \partial B(R)) .$$

Since $n \leq q + 2$ and $v \in H^{1,q}(B(R))$ is a solution of the basic system (1), then, because of the Theorem 1.II in [5], we have, $\forall t \in (0, 1)$,

(12) $$\int_{B(x^0, td)} \| v \|^q dx \leq ct^n \left\{ \int_{B(x^0, d)} \| v \|^q dx + d^q \int_{B(x^0, d)} \| W(Dv) \|^2 dx \right\}$$

where c does not depend on t and x^0.

Moreover, since $n \leq q + 2$ and v is the solution of the Dirichlet problem (5), we have the following estimate for v:

(13) $$\| W(Dv) \|_{L^{2,n-q}(B(R))} \leq c \| W(Du) \|_{L^{2,n-q}(B(R))}$$

(see Theorem 1.I in [5]).

Then, using the Poincaré inequality and account being taken of (13), we have:

(14) $$\int_{B(x^0, d)} \| v \|^q dx \leq c \int_{B(R) \cap B(y^0, 2d)} d^q \| D(v - u) \|^q + \| u(x) \|^q dx \leq$$
$$cd^n \{ \| W(Du) \|^2_{L^{2,n-q}(B(R))} + \sup_{B(R)} \| u \|^q \}$$

and likewise

(15) $$d^q \int_{B(x^0, d)} \| W(Dv) \|^2 dx \leq cd^n \| W(Dv) \|^2_{L^{2,n-q}(B(R))} \leq$$
$$\leq cd^n \| W(Du) \|^2_{L^{2,n-q}(B(R))}$$

From (12), (14) and (15), we eventually deduce that, $\forall t \in (0,1)$ and $\forall x^0 \in \mathrm{B}(\mathrm{R})$

$$\int_{\mathrm{B}(x^0, td)} \| v \|^q \, dx \leq c \, \{ \| \mathrm{W}(\mathrm{D}u) \|^2_{\mathrm{L}^{2,n-q}(\mathrm{B}(\mathrm{R}))} + \sup_{\mathrm{B}(\mathrm{R})} \| u \|^q \} \tag{16}$$

where c does not depend on t and x^0.

Hence, the estimate (11) is proved.

2. Let Ω be a bounded open set in R^n. We consider the system

$$- \sum_i \mathrm{D}_i \, a^i (\mathrm{D}u) = \mathrm{B}(\mathrm{D}u) \qquad \text{in} \quad \Omega \tag{17}$$

where the vectors $a^i(p) \in \mathrm{R}^\mathrm{N}$ satisfy the conditions $a^i(0) = 0$ and (2), (3), whereas $\mathrm{B}(p)$ is a vector of R^N having natural growth. This means that the vector $\mathrm{B}(p)$ is continuous in respect of p and there exist two positive constants, say c and b, such that

$$\| \mathrm{B}(p) \| \leq c + b \, \| \mathrm{W}(p) \|^2 , \qquad \forall p \in \mathrm{R}^{n\mathrm{N}} . \tag{18}$$

As usual, we shall say that $u \in \mathrm{H}^{1,q} \cap \mathrm{L}^\infty(\Omega)$ is a solution of the system (17) if, $\forall \varphi \in \mathrm{H}^{1,q}_0 \cap \mathrm{L}^\infty(\Omega)$,

$$\int_\Omega \sum_i (a^i(\mathrm{D}u) \mid \mathrm{D}_i \varphi) \, dx = \int_\Omega (\mathrm{B}(\mathrm{D}u) \mid \varphi) \, dx \tag{19}$$

Now, we may prove the following theorem

THEOREM 2. *Under the conditions* (2), (3) *and* (18), *if* $u \in \mathrm{H}^{1,q} \cap \mathrm{L}^\infty(\Omega)$ *is a solution of the system* (19), *and*

$$b \cdot \sup_\Omega \| u(x) \| < \nu \tag{20}$$

then, for every ball $\mathrm{B}(\mathrm{R}) \subset\subset \Omega$

$$\| \mathrm{W}(\mathrm{D}u) \|^2_{\mathrm{L}^{2,n-q}(\mathrm{B}(\mathrm{R}))} \leq \frac{c(\nu, \mathrm{M})}{(\nu - b \sup_\Omega \| u \|)^q} \sup_\Omega \| u \|^q \tag{21}$$

Proof. For the sake of simplicity, let us set

$$\mathrm{K} = \sup_\Omega \| u(x) \|$$

Let $B(2\sigma)$ be a ball $\subset \Omega$ and $\theta(x) \in C_0^\infty(R^n)$ be a function having the following properties:

$$(22) \quad 0 \leq \theta \leq 1 \;,\; \theta = 1 \text{ in } B(\sigma) \;,\; \theta = 0 \text{ in } R^n \setminus B(2\sigma) \;,\; \| D\theta \| \leq c\sigma^{-1}.$$

In (19) we assume

$$\varphi = \theta^q u$$

and we obtain

$$\int_\Omega \theta^q \sum_i (a^i(Du) \mid D_i u)\, dx =$$

$$= -q \int_\Omega \sum_i (a^i(Du) \mid \theta^{q-1} D_i \theta \cdot u)\, dx + \int_\Omega (B(Du) \mid \theta^q u)\, dx = A + B.$$

From the condition (3) of strong ellipticity, we deduce that

$$(23) \qquad \nu \int_\Omega \theta^q \| W(Du) \|^2 dx \leq A + B$$

Moreover, $\forall \varepsilon > 0$

$$(24) \qquad \begin{aligned} |A| &\leq c(q, M) \int_\Omega \theta^{q-1} \| D\theta \| V^{q-2}(Du) \| Du \| \cdot \| u \| dx \leq \\ &\leq \varepsilon \int_\Omega \theta^q \| W(Du) \|^2 dx + \frac{c(q, M)}{\varepsilon} \int_\Omega \theta^{q-2} \| D\theta \|^2 V^{q-2}(Du) \| u \|^2 dx \end{aligned}$$

Finally, because of (18),

$$(25) \qquad \begin{aligned} |B| &\leq K \int_\Omega \theta^q (c + b \| W(Du) \|^2)\, dx \leq \\ &\leq bK \int_\Omega \theta^q \| W(Du) \|^2 dx + cK \int_\Omega \theta^q dx \end{aligned}$$

Account being taken of the hypothesis (20) and choosing ε small enough, it follows, from (23), (24), (25), that

$$\int_\Omega \theta^q \, \| W(Du) \|^2 \, dx \leq \tag{26}$$

$$\leq \frac{c(q, M)}{\sigma^2} \frac{K^2}{(\nu - bK)^2} \int_\Omega \theta^{q-2} V^{q-2}(Du) \, dx + c \frac{K}{\nu - bK} \int_\Omega \theta^q \, dx$$

Therefore, by adding the integral

$$\int_\Omega \theta^q V^{q-2}(Du) \, dx$$

to the left-hand side of (26), by the fact that $\theta \leq 1$ and $V(Du) \geq 1$, from (26) we get:

$$\int_\Omega \theta^q V^q(Du) \, dx \leq$$

$$\leq c \left(\int_\Omega \theta^q V^q(Du) \, dx \right)^{1-\frac{q}{2}} \left(\sigma^{\frac{q}{2}\left(\frac{n}{q}-1\right)} \cdot \frac{K^2}{(\nu - bK)^2} + \sigma^{\frac{2n}{q}} \cdot \frac{K}{\nu - bK} \right)$$

and so

$$\int_\Omega \theta^q V^q(Du) \, dx \leq c \, \sigma^{n-q} \left\{ \left(\frac{K}{\nu - bK} \right)^q + \sigma^q \left(\frac{K}{\nu - bK} \right)^{q/2} \right\} \tag{27}$$

From (27) we deduce that, $\forall B(2\sigma) \subset\subset \Omega$ with

$$\sigma \leq \left(\frac{K}{\nu - bK} \right)^{1/2}$$

we have

$$\int_{B(\sigma)} V^q(Du) \, dx \leq c \left(\frac{K}{\nu - bK} \right)^q \sigma^{n-q} \tag{28}$$

From this, the estimate (21) easily follows.

3. The Theorem 2 enables us to improve the estimate (11).

Let Ω be a bounded open set in R^n and $B(R)$ an open ball $\subset\subset \Omega$.

THEOREM 3. *If* $u \in H^{1,q} \cap L^{\infty}(\Omega)$ *is a solution, in* Ω, *of the system* (17), *subjected to the conditions* (2), (3), (8), (18), (20), *and* $v \in H^{1,q}(B(R))$ *is the solution, in* B (R), *of the Dirichlet problem* (5), *then* $v \in L^{\infty}(B(R))$ *and*

$$\sup_{\Omega} \| v \| \leq c(q, M) \left(1 + \frac{1}{\nu - b \sup_{\Omega} \| u \|}\right) \cdot \sup_{\Omega} \| u \|. \tag{29}$$

In fact, the estimate (29) for the vector v is a straightforward consequence of (11) and (21).

In its turn, theorem 3 allows us to define precisely the Hausdorff measure $H_{\beta}(\Omega_0)$, where Ω_0 is the singular set of the vector u, namely:

THEOREM 4. *There exist* $\lambda(\nu, M, n)$, *with* $2 \leq \lambda \leq n$ (1), *and* $t_0 > 1$, *such that, when the conditions* (2), (3), (18), (20) (2) *and* $2 \leq n \leq q + 2$ *hold, if* $u \in\in H^{1,q} \cap L^{\infty}(\Omega)$ *is a solution of the system* (17), *then* u *is partially* α*-Hölder continuous in* Ω, $\forall \alpha < 1 - (n - \lambda)/q$.

If Ω_0 *is the singular set of* u

$$\Omega_0 = \left\{ x^0 \in \Omega : \liminf_{\sigma \to 0} \sigma^{q-n} \int_{B(x^0,\sigma)} \| Du \|^q \, dx > 0 \right\} \tag{30}$$

then Ω_0 *is closed in* Ω *and*

$$H_{n-qt_0}(\Omega_0) = 0 \tag{31}$$

(See the § 5 in [6] for the case $q = 2$).

Finally, note that the results of the theorems 2 and 4 hold also for the solutions $u \in H^{1,q} \cap L^{\infty}(\Omega)$ of the more general systems

$$-\sum_i D_i a^i(x, u, Du) = -\sum_i D_i B^i(x, u) + B(x, u, Du) \tag{32}$$

where $a^i(x, u, p)$ and $B(x, u, p)$ are vectors of R^N which again satisfy the conditions (2), (3), (18) with

$$2b \sup_{\Omega} \| u(x) \| < \nu$$

The $B^i(x, u)$ are vectors of R^N such that

$$\| B^i(x, u) \| \leq c \quad , \quad \forall x \in \Omega \quad \text{and} \quad \forall u \in R^N \tag{33}$$

where the constant c may depend also on $\sup_{\Omega} \| u(x) \|$.

(1) See (3.9) and (3.10) in [5].

(2) For this Theorem it is necessary that $2b \cdot \sup_{\Omega} \| u \| < \nu$.

The dependence of the vectors a^i also on x and u, only requires that one should add a uniform continuity assumption of the following type:

There exists, on $\sigma \geq 0$, a function $\omega(\sigma)$, which is non-decreasing, bounded, continuous, concave and with $\omega(0) = 0$, such that, $\forall x, y \in \Omega$, $\forall u, v \in R^N$ and $\forall p \in R^{nN}$ we have:

$$\sum_i \| a^i(x, u, p) - a^i(y, v, p) \| \leq \omega(\| x - y \|^q + \| u - v \|^q) V^{q-2}(p) \| p \| \tag{34}$$

(See the § 6 in [6] for the case $q = 2$).

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