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A note on complex interpolation of Banach lattices

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Analisi funzionale. — *A note on complex interpolation of Banach lattices.* Nota [*] di Anita Tabacco Vignati [**] e Marco Vignati [**], presentata dal Corrisp. E. Vesentini.

Abstract. — We complete a result of Hernandez on the complex interpolation for families of Banach lattices.

Key words: Banach lattices; Complex interpolation; Concave function.

Riassunto. — *Una nota sull'interpolazione complessa di reticoli di Banach.* Si completa un risultato di E. Hernandez sulla interpolazione complessa di famiglie di reticoli di Banach.

Introduction

In recent years several authors have studied the problem of characterizing the spaces obtained by the complex method of interpolation for families of Banach spaces introduced by R. Coifman, M. Cwickel, R. Rochberg, Y. Sagher and G. Weiss ([CCRSW]).

Many applications follow from the case when the spaces involved are Banach lattices of functions.

In this paper we use properties of concave functions to obtain the reverse of an inclusion proved by E. Hernandez ([Her]), and thus complete the characterization of the Banach lattices obtained by interpolation.

This result is contained in the second author's doctoral dissertation [Vi].
We are grateful to Professors Richard Rochberg and Guido Weiss for introducing us to the subject, and for the advice given during our graduate studies.

1. Some facts on the interpolation of Banach lattices

1.1. A Banach lattice of functions on a σ-finite measure space $(M, d\mu)$ is a subclass X of the class of **C**-valued measurable functions on M, equipped with a norm $\|\cdot\|_X$ so that $(X, \|\cdot\|_X)$ is a Banach space and:

"if g is measurable on M, $f \in X$, and $|g(x)| \leq |f(x)|$ a.e., then $g \in X$ and $\|g\|_X \leq \|f\|_X$".

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A Banach lattice X has the Dominated Convergence Property if:

"given $f \in X$, and $\{f_n\}$ such that $|f_n(x)| \leq |f(x)|$ a.e., and $f_n(x) \to 0$ a.e. as $n \to \infty$, then $\|f_n\|_X \to 0$".

1.2. We write $\varphi \in \phi$ if $\varphi \colon \mathbf{R}^+ \to \mathbf{R}$ is increasing, concave, and $\varphi(0) = 0$. If X is a Banach lattice on M, and $\varphi : M \times [0, \infty) \to \mathbf{R}$ has, for each $x \in M$, $\varphi(x, \cdot) \in \phi$, it is possible to define the class $\varphi(X)$ of all the measurable functions g on M for which there exist $\lambda > 0$ and $f \in X$, $\|f\|_X \leq 1$, such that

$$|g(x)| \leq \lambda \varphi(x, |f(x)|) \qquad \text{on } M. \tag{1.3}$$

Letting

$$\|g\|_{\varphi(X)} = \inf\{\lambda > 0 \,;\, (1.3) \text{ holds}\} \tag{1.4}$$

the space $(\varphi(X), \|\cdot\|_{\varphi(X)})$ is a Banach lattice on M.

General references for Banach lattices can be found in [Cal].

1.5. Let D be the open unit disk in $\mathbf{C}$, and ∂D its boundary. If $\{X_{e^{i\theta}}\}$ is an interpolation family of Banach spaces assigned on ∂D, and every $X_{e^{i\theta}}$ is a Banach lattice on M, we define, for every $z \in D$, the class $[X_{e^{i\theta}}]^z$ of measurable functions f on M for which there exist $\lambda > 0$ and $F : \partial D \times M \to \mathbf{R}$ satisfying $\|F(e^{i\theta}, \cdot)\|_{X_{e^{i\theta}}} \leq 1$ and:

$$|f(x) \leq \lambda \exp \int_0^{2\pi} \log |F(e^{i\theta}, x)| P_z(\theta)\, d\theta \qquad \text{a. e.} \qquad \text{on } M \tag{1.6}$$

where $P_z(\theta) = \dfrac{1}{2\pi} \dfrac{1 - |z|^2}{|z - e^{i\theta}|^2}$.

Letting $\|f\|^z = \inf\{\lambda > 0 \colon (1\ 6) \text{ holds}\}$, the space $([X_{e^{i\theta}}]^z, \|\cdot\|^z)$ becomes a Banach lattice on M.

1.7. E. Hernandez ([Her]) showed that the interpolation space $\{X_{e^{i\theta}}\}[z]$, obtained with the method of [CCRSW], is always continuously embedded in $[X_{e^{i\theta}}]^z$. Moreover, the two spaces coincide, with equality of norms, if $[X_{e^{i\theta}}]^z$ has the Dominated Convergence Property.

2. The main result

When the family of Banach lattices $\{X_{e^{i\theta}}\}$ consists of spaces obtained from a fixed Banach space X, using the method described in 1.2 (as in several applications, including the L^p spaces), the characterization of the interpolation spaces becomes much easier.

We let X be a Banach lattice on M, and $\{\varphi_\theta\}$, $0 \leq \theta < 2\pi$, a family of real-valued functions defined on $M \times [0, \infty)$, such that each $\varphi_\theta(x, \cdot) \in \phi$.

We also assume:

$$\varphi_z(x, t) \equiv \exp \int_0^{2\pi} \log \varphi_\theta(x, t) \, P_z(\theta) \, d\theta < \infty . \tag{2.1}$$

It is then immediate to see that $\varphi_z(z, \cdot) \in \phi$, and it is natural to study the relations between the spaces $[\varphi_\theta(X)]^z$ and $\varphi_z(X)$. E. Hernandez has shown that $\varphi_z(X) \to [\varphi_\theta(X)]^z$ is a norm-decreasing embedding. We can reverse this result, and prove:

THEOREM 2.2. *The Banach lattices $\varphi_z(X)$ and $[\varphi_\theta(X)]^z$ coincide, with equivalence of norms. Moreover, the constants of equivalence do not depend on z.*

Proof. All that we need to prove is that there exists $c > 0$ such that

$$\| f \|_{\varphi_z(X)} \leq c \, \| f \|^z \tag{2.3}$$

for every $f \in [\varphi_\theta(X)]^z$.

If $\varepsilon > 0$ is fixed, and $f \in [\varphi_\theta(X)]^z$, we can find $F : \partial D \times M \to \mathbf{R}$ such that

$$\| F(e^{i\theta}, \cdot) \|_{\varphi_\theta(X)} \leq 1 \text{ and } | f(x) | \leq (1 + \varepsilon) \| f \|^z \exp \int_0^{2\pi} \log | F(e^{i\theta}, x) | \, P_z(\theta) \, d\theta.$$

Using (1.4), for each θ we can find $G_{e^{i\theta}} \in X$ such that $\| G_{e^{i\theta}} \|_X \leq 1$ and $| F(e^{i\theta}, x) | \leq (1 + \varepsilon) \varphi_\theta(x, | G_{e^{i\theta}}(x) |)$ on M.

Thus:

$$\begin{aligned} | f(x) | &\leq (1 + \varepsilon) \| f \|^z \exp \int_0^{2\pi} \log | F(e^{i\theta}, x) | \, P_z(\theta) \, d\theta \\ &\leq (1 + \varepsilon) \| f \|^z \exp \int_0^{2\pi} \log [(1 + \varepsilon) \varphi_\theta(x, | G_{e^{i\theta}}(x) |)] \, P_z(\theta) \, d\theta \\ &\leq (1 + \varepsilon)^2 \| f \|^z \exp \int_0^{2\pi} \log \varphi_\theta(x, | G_{e^{i\theta}}(x) |) \, P_z(\theta) \, d\theta . \end{aligned} \tag{2.4}$$

In order to conclude the proof, we need:

LEMMA 2.5. *Let $\{\varphi_\theta\}$ be a collection of functions in ϕ. For every $t : [0, 2\pi) \to [0, \infty)$ we have:*

$$\exp \int_0^{2\pi} \log \varphi_\theta(t(\theta)) \, P_z(\theta) \, d\theta \leq e^{1/e} \varphi_z(t(z))$$

where φ_z is defined in (2.1) *and* $t(z) = \int_0^{2\pi} t(\theta) \, P_z(\theta) \, d\theta$.

Proof of Lemma 2.5. Let $A = \{\theta : t(\theta) \geq t(z)\}$, $B = [0, 2\pi) \setminus A$, and $|A|_z = \int_A P_z(\theta)\, d\theta$. Since each φ_θ is concave and increasing, we can use Jensen's inequality to obtain:

$$\exp \int_0^{2\pi} \log \varphi_\theta(t(\theta))\, P_z(\theta)\, d\theta$$

$$\leq \left\{ \exp \int_A \log \left[\frac{t(\theta)}{t(z)} \varphi_\theta(t(z)) \right] P_z(\theta)\, d\theta \right\} \left\{ \exp \int_B \log \varphi_\theta(t(z))\, P_z(\theta)\, d\theta \right\}$$

$$\leq \left\{ \exp \int_0^{2\pi} \log \varphi_\theta(t(z))\, P_z(\theta)\, d\theta \right\} \left\{ \exp \int_A \log \frac{t(\theta)}{t(z)} P_z(\theta) \frac{d\theta}{|A|_z} \right]^{|A|_z}$$

$$\leq \varphi_z(t(z)) \left\{ \int_A \frac{t(\theta)}{t(z)} P_z(\theta) \frac{d\theta}{|A|_z} \right\}^{|A|_z}$$

$$\leq \varphi_z(t(z))\, |A|_z^{-|A|_z} \leq e^{1/e} \varphi_z(t(z))\,.$$

Returning to (2.4), and using this last result, we obtain:

$$|f(x)| \leq (1+\varepsilon)^2 \|f\|^z e^{1/e} \varphi_z\left(x, \int_0^{2\pi} |G_{e^{\theta}}(x)|\, P_z(\theta)\, d\theta\right).$$

Setting $G_z(x) \equiv \int_0^{2\pi} |G_{e^{i\theta}}(x)|\, P_z(\theta)\, d\theta$, we have

$$\|G_z\|_X = \left\| \int_0^{2\pi} |G_{e^{i\theta}}|\, P_z(\theta)\, d\theta \right\|_X \leq \int_0^{2\pi} \|G_{e^{i\theta}}\|_X\, P_z(\theta)\, d\theta \leq 1$$

Thus $f \in \varphi_z(X)$, $\|f\|_{\varphi_z(X)} \leq (1+\varepsilon)^2 e^{1/e} \|f\|^z$. Since ε was arbitrarily chosen, the proof is complete.

Remarks. *a*) we obtain the constant $c = e^{1/e} \cong 1.44$; we do not know if, in the general case, the theorem holds with $c = 1$.

b) Theorem 2.2 can be extended to the case of complex interpolation for families of quasi-Banach lattices. This subject is part of the first author's doctoral dissertation [T-V 1], where a complex theory of interpolation for families of quasi-Banach spaces is developed (see also [T-V 2]).

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