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Giovanni Battista Rizza

On a Bianchi-type identity for the almost hermitian manifolds

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Geometria differenziale. — *On a Bianchi-type identity for the almost hermitian manifolds* (*). Nota (**) di GIOVANNI BATTISTA RIZZA, presentata dal Socio E. MARTINELLI.

ABSTRACT. — Almost hermitian manifolds, whose Riemann curvature tensor satisfies an almost complex Bianchi-type identity, are considered. Some local and global theorems are proved. The special cases of parakähler manifolds and of Kähler manifolds are examined.

KEY WORDS: Almost hermitian manifolds; Sectional and bisectional curvatures; Schur-type theorems.

RIASSUNTO. — *Una identità di tipo Bianchi per le varietà quasi hermitiane.* Si considerano varietà quasi hermitiane il cui tensore di curvatura di Riemann soddisfa una identità quasi complessa di tipo Bianchi. Per tali varietà si dimostrano alcuni teoremi locali e globali e si esaminano i casi speciali delle varietà parakähleriane e kähleriane.

1. INTRODUCTION

An investigation about the existence of suitable curvature tensors on an almost hermitian manifold M leads us to consider a special identity for the Riemann curvature tensor R of M (Sec. 2). This identity, involving the almost complex structure J of M, can be regarded as a Bianchi-type identity. If M is a parakähler manifold (in particular, a Kähler manifold), then the identity is satisfied.

In the present paper, assuming first that the above identity is satisfied at a point x of the almost hermitian manifold M, we obtain two local results and derive some consequences (Theorem 1, Theorem 2, Corollary 1, Corollary 2 of Sec. 3). Both theorems assume that M has constant holomorphic curvature at x. The first one concerns a suitable mean of bisectional curvatures; the second the Ricci tensors and the scalar curvatures.

Furthermore, we consider the case when the identity is satisfied at any point x of M. Starting from Theorem 1, Corollary 1, Corollary 2, we immediately derive some global results of Schur type (Theorem 3, Theorem 4, Theorem 5, Theorem 6 of Sec. 5).

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All the theorems of Sec. 3,5 generalize (and sometimes improve) some basic results, known for Kähler manifolds (Sec. 6).

2. A curvature identity

Let M be an almost hermitian manifold of dimension $2m \geq 4$ and of class C^∞. Let g be the metric and J be the almost complex structure of M. All the tensor fields occurring in the paper are assumed to be of class C^∞. For general references, see S. Kobayashi–K. Nomizu [1].

Let x be a point of M and T_x the tangent space to M at x.

A tensor Q of $T_x^* \otimes T_x^* \otimes T_x^* \otimes T_x^*$ is called a *curvature tensor* if and only if Q satisfies

$$\text{(1)} \qquad Q(X, Y, Z, W) = -Q(Y, X, Z, W)$$

$$\text{(2)} \qquad Q(X, Y, Z, W) = Q(Z, W, X, Y)$$

$$\text{(3)} \qquad Q(X, Y, Z, W) + Q(X, Z, W, Y) + Q(X, W, Y, Z) = 0$$

for any X, Y, Z, W of T_x (F. Tricerri–L. Vanhecke [9], p. 367).

A tensor Q of $T_x^* \otimes T_x^* \otimes T_x^* \otimes T_x^*$ is called a *Kähler curvature tensor* if and only if Q satisfies (1), (2), (3) and

$$\text{(4)} \qquad Q(X, Y, Z, W) = Q(X, Y, JZ, JW)$$

for any X, Y, Z, W of T_x.

It is well known that the classical *Riemann tensor* R satisfies identities (1) (2), (3). It is also known that if M is a *parakähler manifold* (G.B. Rizza [3]), i.e. an F*-space* (S. Sawaki [7]), in particular a *Kähler manifold,* then R satisfies also identity (4).

Since for a general almost hermitian manifold R *does not satisfy* (4), starting from R, we try to construct a new tensor satisfying (1), (2), (3), (4). So we consider the tensor P of $T_x^* \otimes T_x^* \otimes T_x^* \otimes T_x^*$ defined by

$$\text{(5)} \qquad \begin{aligned} 4\,P(X, Y, Z, W) &= R(X, Y, Z, W) + R(X, Y, JZ, JW) \\ &\quad + R(JX, JY, Z, W) + R(JX, JY, JZ, JW). \end{aligned}$$

It is immediate that if M is a parakähler manifold (a Kähler manifold), then P reduces to the Riemann tensor R. It is also easy to check that P *satisfies identities* (1), (2), (4).

A necessary and sufficient condition in order that P *satisfies also identity* (3) *is that the Riemann tensor* R *satisfies the identity*

$$(*)\quad \begin{aligned} &R(X, Y, JZ, JW) + R(X, Z, JW, JY) + R(X, W, JY, JZ) \\ &+ R(JX, JY, Z, W) + R(JX, JZ, W, Y) + R(JX, JW, Y, Z) = 0 . \end{aligned}$$

The proof is elementary. It is worth remarking that, if M is assumed to be a parakähler manifold (a Kähler manifold), then R satisfies (4) and the identity (*) reduces simply to the first Bianchi identity. So we may regard (*) as a Bianchi-type identity.

3. LOCAL RESULTS

In this Section we assume that M *is an almost hermitian manifold, whose Riemann tensor* R *satisfies identity* (*) *at the point* x.

Let p, q, r, s be 2-dimensional oriented subspaces of the tangent vector space T_x (oriented planes of T_x). We denote by χ_{pq}, K_r, δ_r the *bisectional curvature* for the couple p, q, the *sectional curvature* (*riemannian curvature*) for the plane r, the *holomorphic deviation* of r (see for istance G.B. Rizza [5], [6]).

Let ρ, $\mathring{\rho}$ be the *Ricci tensor*, the *hermitian Ricci tensor* at point x and τ, $\mathring{\tau}$ the *scalar curvature*, the *hermitian scalar curvature* of M at point x (see for istance G.B. Rizza [3]).

We will prove the following results

THEOREM 1. *If* M *has constant holomorphic curvature* c *at* x, *then for any couple* p, q *of oriented planes of* T_x, *we have*

$$(6)\qquad \chi_{pq} + \chi_{pJq} + \chi_{Jpq} + \chi_{JpJq} = c\,(\cos pq + \cos p\,J\,q + 2\cos\delta_p \cos\delta_q)$$

and consequently (*for* $q = p$)

$$(7)\qquad K_p + 2\,\chi_{pJp} + K_{Jp} = c\,(1 + 3\cos^2\delta_p)\,.$$

THEOREM 2. *If* M *has constant holomorphic curvature* c *at* x, *then for any couple of vectors* Y, W *of* T_x *we have*

$$(8)\quad \rho(Y, W) + \rho(JY, JW) + \mathring{\rho}(Y, JW) + \mathring{\rho}(W, JY) = 2\,c\,(m+1)\,g(Y, W)$$

$$(9)\qquad \tau + \mathring{\tau} = 2\,c\,m\,(m+1)\,.$$

We may note that, starting from a couple of oriented planes p, q and using the almost complex structure J, we are led to introduce the system $S_{pq}(J)$, formed by the couples p, q; p, Jq; Jp, q; Jp, Jq. It is worth remarking that the system $S_{pq}(J)$ is J-*invariant.* Now, the expression at first member of (6), divided by 4, can be regarded as a *mean of the bisectional curvatures of the couples* of $S_{pq}(J)$. Similarly the first member of (7), divided by 4, appears as a *mean of the bisectional curvatures of the couples of* $S_{pp}(J)$.

From Theorem 1 we derive some consequences

COROLLARY 1. *Under the assumptions of Theorem* 1, *for any couple* h_1, h_2 *of canonically oriented holomorphic planes of* T_x *we have*

$$\chi_{h_1 h_2} = \frac{c}{2}(1 + \cos h_1 h_2) = c \cos^2 \frac{1}{2} h_1 h_2 . \tag{10}$$

In particular, if M *has constant biholomorphic curvature at point* x, *then this constant is zero.*

COROLLARY 2. *If* M *has constant curvature* C *at point* x, *then* $C = 0$.

4. PROOFS

To prove Theorem 1, note first that, since R satisfies (*) at the point x, then P satisfies the identities (1), (2), (3), (4) at x (Sec. 2). Remark also that we have

$$P(X, JX, X, JX) = R(X, JX, X, JX) \tag{11}$$

Consider now the tensor R_0 of $T_x^* \otimes T_x^* \otimes T_x^* \otimes T_x^*$ defined by

$$\begin{aligned} 4 R_0(X, Y, Z, W) &= g(X, Z) g(Y, W) - g(X, W) g(Y, Z) \\ &+ 2 g(X, JY) g(Z, JW) + g(X, JZ) g(Y, JW) - g(X, JW) g(Y, JZ) \end{aligned} \tag{12}$$

and note also that, at the point x, R_0 satisfies the identities (1), (2), (3), (4) of Sec. 2 and also the identity

$$R_0(X, JX, X, JX) = g(X, X) g(X, X) \tag{13}$$

([1], vol. 2, p. 167).

On the other hand, from the assumption that M has constant holomorphic curvature c at the point x, we immediately derive

$$R(X, JX, X, JX) = c R_0(X, JX, X, JX) . \tag{14}$$

Therefore, from (11), (14) we get

$$P(X, JX, X, JX) = c R_0(X, JX, X, JX) \tag{15}$$

for any X of T_x.

Finally, using Proposition 7.1 at p. 166 of [1], vol. 2, we obtain

$$P(X, Y, Z, W) = c R_0(X, Y, Z, W) \tag{16}$$

for any X, Y, Z, W of T_x. Taking now into account the definitions of χ_{rs}, $\cos rs$, δ_t ([5], [6], Sec. 2), we can write (16) in the form (6). In particular, since $\cos p\, Jp = \cos^2 \delta_p$ ([6], Sec. 2), if $q = p$ then equation (6) reduces to equation (7). So Theorem 1 is completely proved.

We consider now Corollary 1. If h_1, h_2 are canonically oriented holomorphic planes, that is if $h_1 = Jh_1$, $h_2 = Jh_2$ and $\delta_{h_1} = \delta_{h_2} = 0$ ([6], Sec. 2), then equation (6) reduces simply to equation (10). This proves the first part. If M has constant biholomorphic curvature c at the point x, then the assumption of Theorem 1 is obviously satisfied and equation (10) reduces simply to $c(1\text{-}\cos h_1 h_2) = 0$ for any couple h_1, h_2 of canonically oriented holomorphic planes of T_x. Since $\dim M \geq 4$, there exist in T_x mutually orthogonal planes h_1, h_2. This implies $c = 0$ and the proof of Corollary 1 is complete.

Now let a be an antiholomorphic oriented plane of T_x; so a is orthogonal to Ja and $\delta_a = \frac{\pi}{2}$ ([6], Sec. 2). From the assumption of Corollary 2 we derive $\chi_{a\,Ja} = C \cos a\, Ja = 0$ ([5], Theorem 1). On the other side we can use Theorem 1. Considering equation (7) for $p = a$, we come immediately to the end of the proof of Corollary 2.

Finally, we prove Theorem 2. As we have seen, from the assumption we can derive equation (16), where P and R_0 are defined by (5), (12) respectively.

Since we have

$$R(X, Y, Z, W) = g(R(Z, W)Y, X) = g(J(R(Z, W)Y), JX)$$

we can write

$$\begin{aligned} R(Z, W)Y + R(JZ, JW)Y - J(R(Z, W)JY) - J(R(JZ, JW)JY) = \\ = c[g(Y, W)Z - g(Y, Z)W + g(Y, JW)JZ - g(Y, JZ)JW] + \\ + 2c\, g(Z, JW)JY \end{aligned} \tag{17}$$

for any Y, Z, W of T_x.

We recall now that the Ricci tensor ρ and the hermitian Ricci tensor $\mathring{\rho}$ at the point x are defined by

$$\begin{aligned} \rho(Y, W) &= \mathit{trace}\, Z \mapsto R(Z, W)Y = \mathit{trace}\, Z \mapsto -J(R(JZ, W)Y) \\ \mathring{\rho}(Y, W) &= \mathit{trace}\, Z \mapsto R(JZ, W)Y = \mathit{trace}\, Z \mapsto J(R(Z, W)Y). \end{aligned} \tag{18}$$

It is worth remarking that we have

$$\rho(Y, W) = \rho(W, Y), \quad \mathring{\rho}(Y, W) = -\mathring{\rho}(W, Y) \tag{19}$$

for any Y, W of T_x ([3], p. 5).

Let ω_1, ω_2 be the homomorphisms of T_x into itself, mapping Z into the first, the second member of (17), respectively. Since $\omega_1 = \omega_2$, we have *trace* $\omega_1 =$ *trace* ω_2.

Now, taking into account definitions (18) and equation (19), we see that *trace* ω_1 reduces to the first member of (8). Similarly, since we have *trace* $I = 2m$ ($I =$ identity), *trace* $J = 0$, and

$$\textit{trace}\colon Z \mapsto g(Z, Y)\, W = g(Y, W) = g(JY, JW)$$

we see that *trace* ω_2 reduces to the second member of (8). So equation (8) is proved.

Finally, consider the vectors $\rho(W)$, $\mathring{\rho}(W)$ implicitly defined by

$$\rho(Y, W) = g(Y, \rho(W)) = g(JY, J\rho(W)),$$

$$\mathring{\rho}(Y, W) = g(Y, \mathring{\rho}(W)) = g(JY, J\mathring{\rho}(W)).$$

Using (19), from equation (8) we derive

$$\rho(W) - J\rho(JW) + \mathring{\rho}(JW) + J\mathring{\rho}(W) = 2c(m+1)\, W. \tag{20}$$

We recall now that the *scalar curvature* τ and the *hermitian scalar curvature* $\mathring{\tau}$ at the point x can be defined by

$$\begin{aligned} \tau &= \textit{trace}\colon W \mapsto \rho(W) = \textit{trace}\colon W \mapsto -J\rho(JW) \\ \mathring{\tau} &= \textit{trace}\colon W \mapsto \mathring{\rho}(JW) = \textit{trace}\colon W \mapsto J\mathring{\rho}(W). \end{aligned} \tag{21}$$

Denote by α the homomorphism of T_x in itself, mapping W into the first member of (20). Considering *trace* α, we come immediately to (9). So Theorem 2 is completely proved.

5. Further results

We assume now that M *is an almost hermitian manifold, whose Riemann tensor field satisfies identity* (*) *at any point* x *of* M.

From Corollary 1, Corollary 2 (Sec. 3) we immediately derive

THEOREM 3. *Let the sectional curvature* K_r *be constant for any plane* r *of* T_x. *If this property is true at any point* x *of* M, *then* M *is a flat manifold.*

THEOREM 4. *Let the biholomorphic curvature* $\chi_{h_1 h_2}$ *be constant for any couple of canonically oriented holomorphic planes* h_1, h_2 *of* T_x. *If this property is true at any point* x *of* M, *then* M *has biholomorphic curvature equal to zero.*

A remark in Sec. 3 leads us to call *mean bisectional curvature* $\mathscr{C}_{pq}$ *of the system* $S_{pq}(J)$ the first member of (6) divided by 4. In particular, the mean bisectional curvature of $S_{pp}(J)$ will be denoted by $C_p = \mathscr{C}_{pp}$.

We are now able to state the Theorems

THEOREM 5. *If the absolute value of the mean bisectional curvature* $\mathscr{C}_{pq}$ *is constant for any couple of oriented planes* p, q *of* T_x *and this property is true at any point* x *of* M, *then* $\mathscr{C}_{pq} = 0$ *on* M.

THEOREM 6. *If the mean bisectional curvature* C_p *is constant for any plane* p *of* T_x *and this property is true at any point* x *of* M, *then* $C_p = 0$ *on* M.

It is worth remarking that the constants occurring in the previous theorems, a priori depending on the point x, *do not really depend on* x. So Theorem 3, Theorem 4, Theorem 5, Theorem 6 can be regarded as theorems of Schur-type.

The proofs of Theorem 5, Theorem 6 are easy.

Denote by $\bar{c}$ the constant, at the point x, occurring in the assumption of Theorem 5. Let h be a canonically oriented holomorphic plane of T_x and put $p = q = h$. Since we have $h = J\,h$, we obtain $|K_h| = |\chi_{hh}| = |\mathscr{C}_{hh}| = \bar{c}$. Using continuity, we derive that K_h is constant for any holomorphic plane h T_x; namely $K_h = \bar{c}$ or $K_h = -\bar{c}$. So we are able to use Theorem 1 of Sec. 3. Since dim $M \geq 4$, there exist in T_x a couple h_1, h_2 of orthogonal canonically oriented holomorphic planes. From $h_1 = J\,h_1$, $h_2 = J\,h_2$, $\delta_{h_1} = \delta_{h_2} = 0$, we derive that the second member of (6) reduces to $2\bar{c}$, $-2\bar{c}$ respectively. Hence we have $\bar{c} = 0$ and Theorem 5 is proved.

Similarly, denote by c the constant, at the point x, occurring in the assumption of Theorem 6 and let h be a canonically oriented holomorphic plane of T_x. Since we have $h = J\,h$, we immediately derive $K_h = C_h = c$. So we can use Theorem 1 of Sec. 3. Consider now an antiholomorphic oriented plane a of T_x. Since $\delta_a = \frac{\pi}{2}$, equation (7) for $p = a$ reduces to $4c = c$. Hence we have $c = 0$ and Theorem 6 is proved.

6. PARAKÄHLER MANIFOLDS. KÄHLER MANIFOLDS

At the end of Sec. 2, we have seen that parakähler manifolds (and in particular Kähler manifolds) are a special case of manifold, satisfying the identity (*) at any point.

Assume now that M is a *parakähler manifold.*

We recall first that in the present assumption, for any couple p, q of oriented planes of T_x we have

$$\chi_{pq} = \chi_{pJq} = \chi_{Jpq} = \chi_{JpJq} \tag{22}$$

and consequently

$$\mathscr{C}_{pq} = \chi_{pq} \quad , \quad C_p = K_p$$

Moreover, for any couple of vectors Y, W of T_x we have

$$\rho(Y, W) = \rho(JY, JW) = \mathring{\rho}(Y, JW) = \mathring{\rho}(W, JY) \; ; \; \tau = \mathring{\tau} \tag{23}$$

([3], Sec. 7, 8).

It is also worth remarking that if M has constant holomorphic curvature $c \neq 0$, then M is a Kähler manifold ([8], Theorem 4.6).

Now, taking into account (22), from Theorem 1 of Sec. 3 we immediately derive

THEOREM 1′. *If* M *is a parakähler manifold of zero holomorphic curvature at* x, *then for any couple* p, q *of oriented plane of* T_x *we have* $\chi_{pq} = 0$.

THEOREM 1″. *If* M *is a Kähler manifold of constant holomorphic curvature* c *at* x, *then for any couple* p, q *of oriented plane of* T_x *we have*

$$\chi_{pq} = \frac{c}{4} (\cos pq + \cos p\, J\, q + 2 \cos \delta_p \cos \delta_q) \tag{24}$$

and consequently (for $q = p$*)*

$$K_p = \frac{c}{4} (1 + 3 \cos^2 \delta_p) \, . \tag{25}$$

We remark explicitly that equation (24), expressing the bisectional curvature χ_{pq} in a simple and elegant way, can be derived also from equation $R = c\, R_0$ ([1] vol. 2, Proposition 7.3, p. 167). We recall also that equation (25) is known ([2], p. 88) and appears also in [1], vol. 2, in a slightly different form (Proposition 7.4, p. 167).

We add here a further remark. If M is a Kähler manifold of constant holomorphic curvature c at x, then, by virtue of (23), Theorem 2 of Sec. 3 reduces to the known fact that M satisfies the *Einstein condition* at the point x (in particular, M has constant scalar curvature at x).

Finally, we consider Theorem 3, Theorem 4, Theorem 5, Theorem 6 of Sec. 5 in the special case when M is a Kähler manifold. We immediately see that by virtue of (22) Theorem 3 and Theorem 6 coincide. We obtain the classical conclusion that M is a flat manifold. Similarly, since we have $\mathscr{C}_{pq} = \chi_{pq}$ and in particular $\mathscr{C}_{rr} = K_r$, from Theorem 5 we come to the same conclusion about M. Under the assumption of Theorem 4, we derive easily that M has zero holomorphic curvature. Therefore, using (24), (25), we find again that M is a flat manifold.

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