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## Converging semigroups of holomorphic maps

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**Geometria.** – *Converging semigroups of holomorphic maps.* Nota di MARCO ABATE, presentata (*) dal Corrisp. E. VESENTINI.

ABSTRACT. – In this paper we study the semigroups $\Phi:\mathbf{R}^+ \to \mathrm{Hol}(D, D)$ of holomorphic maps of a strictly convex domain $D \subset \boldsymbol{C}^n$ into itself. In particular, we characterize the semigroups converging, uniformly on compact subsets, to a holomorphic map $h:D\to \boldsymbol{C}^n$.

KEY WORDS: Semigroups of holomorphic maps; Convex domains; Iteration of holomorphic maps; Fixed points.

RIASSUNTO. – *Semigruppi convergenti di applicazioni olomorfe.* In questa nota vengono caratterizzati quei semigruppi $\Phi:\mathbf{R}^+ \to \mathrm{Hol}(D, D)$ di applicazioni olomorfe di un dominio strettamente convesso $D \subset \boldsymbol{C}^n$ in sé che convergono, uniformemente sui compatti, ad un'applicazione olomorfa $h: D\to \boldsymbol{C}^n$.

In 1926, Wolff and Denjoy (see [4], [8], [9] and [10]) proved the following theorem:

THEOREM 0.1: (Wolff-Denjoy) *Let* $\Delta$ *be the unit disk in the complex plane, and* $f:\Delta\to\Delta$ *a holomorphic function. Then the sequence* $\{f^n\}$ *of iterates of* f *converges, uniformly on compact sets, to a holomorphic function* $h: \Delta\to \mathbf{C}$ *iff* f *is not an automorphism of* $\Delta$ *with exactly one fixed point.*

This very nice result can be generalized in two ways. The first one is increasing the dimension of the ambient space, that is looking at domains in $\mathbf{C}^n$, with $n \geq 1$.

For strictly convex $C^2$ bounded domains in $\mathbf{C}^n$, a complete result is known (see [1]):

THEOREM 0.2: *Let* D *be a strictly convex bounded* $C^2$ *domain, and* $f:D\to D$ *a holomorphic map. Then the sequence* $\{f^n\}$ *of iterates converges, uniformly on compact sets, iff either*
(i) f *has a fixed point* $z_0 \in D$, *and the differential* $df(z_0)$ *has no eigenvalues* $\lambda \neq 1$ *with* $|\lambda| = 1$, *or*
(ii) f *has no fixed points.*

It is worth noticing that, by Schwarz's lemma, if $D = \Delta$ Theorem 0.2 becomes exactly Theorem 0.1.

(*) Nella seduta del 13 febbraio 1988.

The second kind of generalization is changing the object of study. The main property of a sequence of iterates $\{f^n\}$ is that $f^m \circ f^n = f^{m+n}$ for all $m, n \in \mathbf{N}$. So, it is very natural to investigate the properties of a *semigroup* of holomorphic maps in a domain $D \subset \mathbf{C}^n$, that is of a continuous map $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ such that $\Phi_0 = \mathrm{id}_D$, and $\Phi_s \circ \Phi_t = \Phi_{s+t}$, for all $s, t \in \mathbf{R}^+$. In this paper $\mathrm{Hol}(D, D)$ is always endowed with the compact-open topology; by Vitali's theorem (see e.g. [6]), convergence in this topology is equivalent to punctual convergence.

Vesentini, in a series of seminars, characterized the converging semigroups in $\Delta$ (but compare also [3]):

THEOREM 0.3: (Vesentini) *Let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(\Delta, \Delta)$ be a semigroup of holomorphic functions in $\Delta$. Then $\Phi$ converges for $t \to \infty$ to a holomorphic function $h: \Delta \to \mathbf{C}$ iff no $\Phi_t$ is an automorphism of $\Delta$ with exactly one fixed point.*

The aim of this note is to extend Theorem 0.3 to strictly convex domains in $\mathbf{C}^n$, exactly as Theorem 0.2 was a generalization of Theorem 0.1.

Let $D$ be a strictly convex bounded $C^2$ domain, and $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ a semigroup of holomorphic maps in $D$. We shall say that $z_0 \in D$ is a *fixed point* of $\Phi$ if $z_0 \in \mathrm{Fix}(\Phi_t)$ for all $t \in \mathbf{R}^+$, where $\mathrm{Fix}(\Phi_t)$ is the fixed point set of $\Phi_t$. On the other hand, $\Phi$ is *fixed point free* if $\mathrm{Fix}(\Phi_t) = \varnothing$ for all $t > 0$. An important fact we shall show later is that either $\Phi$ has a fixed point, or $\Phi$ is fixed point free.

The first step toward our aim is:

PROPOSITION 1.1: *Let $D$ be a strictly convex bounded $C^2$ domain, and $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ a semigroup in $D$. Assume $\mathrm{Fix}(\Phi_{t_0}) = \varnothing$ for some $t_0 > 0$. Then the semigroup converges to a constant $x \in \partial D$.*

*Proof*: In [1] it is shown that the sequence $\{\Phi_{nt_0}\} = \{(\Phi_{t_0})^n\}$ converges, uniformly on compact sets, to a point $x \in \partial D$.

Fix $z_0 \in D$, and let $K = \{\Phi_s(z_0) \mid 0 \leq s \leq t_0\}$. By continuity, $K$ is a compact subset of $D$; therefore, for all $\epsilon > 0$ there is $n_\epsilon \in \mathbf{N}$ such that

$$n \geq n_\epsilon \Rightarrow \|\Phi_{nt_0} - x\|_K < \epsilon \Rightarrow \sup_{0 \leq s \leq t_0} |\Phi_{nt_0+s}(z_0) - x| < \epsilon,$$

that is $|\Phi_t(z_0) - x| < \epsilon$ for all $t \geq n_\epsilon t_0$. In other words, $\Phi_t(z_0)$ converges to $x$ for all $z_0 \in D$; by Vitali's theorem (cf. [6]), $\Phi_t \to x$, q.e.d.

COROLLARY 1.2: *Let $D$ be a strictly convex bounded $C^2$ domain, and $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ a semigroup in $D$. Then $\Phi$ is fixed point free iff $\mathrm{Fix}(\Phi_{t_0}) = \varnothing$ for some $t_0 > 0$.*

*Proof*: One direction is trivial. Conversely, if $\mathrm{Fix}(\Phi_{t_0}) = \varnothing$ for some $t_0 > 0$, then by Proposition 1.1 the semigroup converges to a point in the boundary of $D$; hence no $\Phi_t$ with $t > 0$ can have a fixed point, q.e.d.

The next step is crucial:

PROPOSITION 1.3: *Let* D *be a strictly convex bounded* $C^2$ *domain, and* $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ *a semigroup in* D. *Assume* $\mathrm{Fix}(\Phi_{t_0}) \neq \varnothing$ *for some* $t_0 > 0$. *Then there is a non-empty closed connected submanifold* F *of* D *contained in* $\mathrm{Fix}(\Phi_t)$ *for every* $t \in \mathbf{R}^+$. *In particular,* $\Phi$ *has fixed points.*

*Proof*: Put $f_n = \Phi_{t_0/2^n}$; then $f_0 = \Phi_{t_0}$ and $(f_{n+1})^2 = f_n$. Let $F_n = \mathrm{Fix}(f_n)$; by Vigué's work (cf. [7]), every $F_n$ is a closed connected submanifold of $D$, and $F_n \supset F_{n+1}$. Moreover, $F_0 \neq \varnothing$; then, by Corollary 1.2 every $F_n$ is not empty.

So we have constructed a decreasing sequence of non-empty closed connected submanifolds of $D$; therefore dim $F_n$ should eventually become constant. But $F_{n+1}$ is a closed submanifold of $F_n$, which is connected; hence $\dim F_{n+1} = \dim F_n$ implies $F_{n+1} = F_n$, and the sequence $\{F_n\}$ itself is eventually constant. Let $F$ be its limit.

By construction, $F \subset \mathrm{Fix}(\Phi_{t_0/2^n})$ for all $n \in \mathbf{N}$; hence $F \subset \mathrm{Fix}(\Phi_{p/2^n})$ for all $p$, $n \in \mathbf{N}$. Since $\{pt_0/2^n \mid p, n \in \mathbf{N}\}$ is dense in $\mathbf{R}^+$, we finally get $F \subset \mathrm{Fix}(\Phi_t)$ for all $t \in \mathbf{R}^+$, q.e.d.

Corollary 1.2 and Proposition 1.3 show, as promised, that a semigroup in a strictly convex domain either has a fixed point or is fixed point free.

Proposition 1.3 is somewhat related to the following result:

THEOREM 1.4: *Let* D *be a convex bounded domain, and* $\mathfrak{F} \subset \mathrm{Hol}(D, D) \cap C^0(\overline{D})$ *a family of commuting holomorphic maps. Then* $\mathfrak{F}$ *has a fixed point, that is there exists* $z_0 \in \overline{D}$ *such that* $f(z_0) = z_0$ *for all* $f \in \mathfrak{F}$.

A proof for two maps (and D smooth) is contained in [1]; the proof of the general statement will appear in [2] and [2a].

Coming back to our problem, let $\Phi: \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ be a semigroup with a fixed point $z_0 \in D$. Then we can associate to $\Phi$ the linear semigroup $A: \mathbf{R}^+ \to \mathbf{GL}(n, \mathbf{C})$ given by

$$A_t = d\Phi_t(z_0).$$

Let $X_\Phi$ be its infinitesimal generator; $X_\Phi$ is called the *spectral generator* at $z_0$ of the semigroup $\Phi$.

Since we are working in a finite dimensional space, $A_t = \exp(tX_\Phi)$ for all $t \in \mathbf{R}^+$. In particular, every eigenvalue of $A_t$ is of the form $e^{t\lambda}$, where $\lambda$ is an eigenvalue of $X_\Phi$. Furthermore (see e.g. [5]), every eigenvalue of $A_t$ is contained in $\overline{\Delta}$, and this is possible iff every eigenvalue of $X_\Phi$ has nonpositive real part.

This is all we need for our main result:

THEOREM 1.5: *Let* D *be a strictly convex bounded* $C^2$ *domain, and* $\Phi : \mathbf{R}^+ \to \mathrm{Hol}(D, D)$ *a semigroup in* D. *Then* $\Phi$ *converges iff either*
(i) $\Phi$ *has a fixed point* $z_0 \in D$, *and the spectral generator at* $z_0$ *of* $\Phi$ *has no nonzero purely imaginary eigenvalues, or*
(ii) $\Phi$ *has no fixed points.*

*Proof*: If (i) holds, then for every $t_0 > 0$ the differential $d\Phi_{t_0}(z_0)$ has no eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$; therefore, by Theorem 0.2, $\{\Phi_{nt_0}\}$ converges. In particular, for every fixed $p \in \mathbf{N}$ the sequence $\{\Phi_{n/2^p}\}$ converges, and the limit does not depend on $p$ – for if $p < q$ then $\{\Phi_{n/2^p}\}$ is a subsequence of $\{\Phi_{n/2^q}\}$. Since $\{n/2^p \mid n, p \in \mathbf{N}\}$ is dense in $\mathbf{R}^+$, this implies that the whole semigroup converges, as we wanted to show.

If (ii) holds, then the semigroup converges by Proposition 1.1.

Conversely, assume $\Phi$ converging. Then either $\Phi$ has no fixed points, or (by Theorem 0.2) every $d\Phi_t(z_0)$ has no eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$, where $z_0 \in D$ is a fixed point of $\Phi$. Hence the spectral generator at $z_0$ of $\Phi$ cannot have nonzero purely imaginary eigenvalues, and we are done, q.e.d.

Again, when $D = \Delta$ Theorem 1.5 becomes Theorem 0.3. Indeed, if $\Phi$ has a fixed point $z_0 \in \Delta$, then $\Phi_t'(z_0) = e^{t\lambda}$, where $\lambda \in \mathbf{C}$ is the spectral generator at $z_0$ of $\Phi$. By Schwarz's lemma, $\mathrm{Re}(\lambda) \leq 0$, and $\mathrm{Re}(\lambda) = 0$ iff every $\Phi_t$ is an automorphism of $\Delta$. Finally, $\lambda = 0$ iff $\Phi_t = \mathrm{id}_\Delta$ for all $t \in \mathbf{R}^+$, and Theorem 0.3 is completely recovered by Theorem 1.5.

We end this note with some counterexamples showing that it is impossible to relax the hypotheses in Theorem 1.5.

Let $D = \{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 + |w|^{-2} < 3\}$. $D$ is a strictly pseudoconvex bounded smooth domain in $\mathbf{C}^2$. A semigroup in $D$ is given by

$$\Phi_t(z, w) = (z, e^{it}w).$$

$\Phi$ is fixed point free, and does not converge.

Let $D = \Delta \times \Delta$ be the bidisk in $\mathbf{C}^2$. $D$ is a convex bounded domain; a semigroup in $D$ is given by

$$\Phi_t(z, w) = \left(e^{it}z, \frac{w + \tanh(t)}{1 + \tanh(t)w}\right).$$

Again, $\Phi$ is fixed point free, and does not converge.

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