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Harmonie reflections

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Geometria differenziale. — *Harmonic reflections.* Nota di LIEVEN VANHECKE e MARIA-ELENA VAZQUEZ-ABAL, presentata (*) dal Socio E. MARTINELLI.

ABSTRACT. — We study local reflections φ_σ with respect to a curve σ in a Riemannian manifold and prove that σ is a geodesic if φ_σ is a harmonic map. Moreover, we prove that the Riemannian manifold has constant curvature if and only if φ_σ is harmonic for all geodesics σ.

KEY WORDS: Harmonic maps; Reflections with respect to a curve; Harmonic reflections.

RIASSUNTO. — *Riflessioni armoniche.* Si studia la riflessione locale φ_σ rispetto ad una curva σ in una varietà riemanniana e si dimostra che σ è una geodetica se φ_σ è un'applicazione armonica. Inoltre si prova che la varietà è a curvatura costante se e solamente se φ_σ è armonica per tutte le geodetiche σ.

1. INTRODUCTION

Local geodesic symmetries on a Riemannian manifold are local diffeomorphisms and the properties of these transformations may be used to characterize several nice classes of Riemannian manifolds. For example, in [3] C. Dodson and the two authors of this paper proved the following: *A Riemannian manifold is locally symmetric if and only if all local geodesic symmetries are harmonic.*

Local geodesic symmetries are local reflections with respect to a point. In [8], [9], [10] this class of transformations has been generalized and the notion of a local reflection with respect to a submanifold, in particular with respect to a curve, has been introduced and studied. These local reflections are local diffeomorphisms. In this paper we consider *harmonic reflections with respect to a curve.* Our aim is to prove

THEOREM 1 - *Let* σ: $[a, b] \rightarrow$ M *be a topologically embedded curve in a Riemannian manifold* M. *If the local reflection with respect to* s *is harmonic, then* s *is a geodesic.*

THEOREM 2 - *Let* (M, g) *be a connected Riemannian manifold. Then* (M, g) *is a space of constant curvature if and only if the local reflections with respect to all geodesics are harmonic.*

(*) Nella seduta del 20 novembre 1987.

2. Harmonic maps

Let (M, g) and (N, h) be two Riemannian manifolds with metrics g and h and let f: (M, g) → (N, h) be a smooth map. The pullback f^*h is a semi-definite symmetric covariant tensor of order two, called the *first fundamental form*. Further, the covariant differential $\nabla(df)$ is a symmetric tensor of order two with values in $f^{-1}(TN)$, called the *second fundamental form* of f (see [4], [5]). A map with vanishing second fundamental form is said to be *totally geodesic*.

The trace of $\nabla(df)$ is denoted by $\tau(f)$ and is called the *tension field* of f. A map with vanishing tension field is called a *harmonic* map.

If $U \subset M$ is a domain with coordinates $(x^1, \ldots, x^m)$ and $V \subset N$ is a domain with coordinates $(y^1, \ldots, y^n)$ such that $f(U) \subset V$, then f can be locally represented by $y^\alpha = f^\alpha(x^1, \ldots, x^m)$, $\alpha = 1, \ldots, n$. The metric tensor g is represented by $g(x) = g_{ij}(x)\,dx^i dx^j$, $i, j = 1, \ldots, m$, and similarly we have $h(y) = h_{\alpha\beta}(y)\,dy^\alpha dy^\beta$, $\alpha, \beta = 1, \ldots, n$. $df(x)$ is represented by the matrix $\left(\dfrac{\partial f^\alpha}{\partial x^i}\right)$. In this case we have

$$(f^*h)_{ij} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}h_{\alpha\beta},$$

$$(\nabla(df))^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}, \tag{1}$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols for (M, g) and (N, h) respectively.

It follows that f is harmonic if and only if

$$\tau(f)^\gamma = g^{ij}(\nabla(df))^\gamma_{ij} = 0. \tag{2}$$

3. Reflections with respect to a curve

Let σ: [a, b] → (M, g) be a smooth embedded curve in a Riemannian manifold (M, g) and denote by U a tubular neighbourhood of σ, i.e.

$U = \{p \in M \mid$ there exists a unique geodesic γ of M through p cutting σ orthogonally$\}$.

Then, for any $p \in U$ we may put

$$p = \exp_{\sigma(t)}(ru), \quad u \in T^\perp_{\sigma(t)}\sigma, \quad \|u\| = 1, \quad t \in [a, b],$$

where $r = d(p, \sigma(t))$.

The map $\varphi_\sigma : U \to U$ *defined by*

$$\varphi_\sigma : p = \exp_{\sigma(t)}(ru) \mapsto \varphi_\sigma(p) = \exp_{\sigma(t)}(ru)$$

is a local diffeomorphism and is called a *local reflection with respect to σ*.

To treat this map φ_σ analytically we use *Fermi coordinates*. We follow the treatment developed in [7], [9]. For the Riemannian manifold (M, g) let ∇ denote the Levi-Civita connection and R the Riemannian curvature tensor, defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields X, Y on M.

First, let σ be a unit speed curve, i.e. $\|\dot{\sigma}\| = 1$, and let $\{e_i, i = 1, \ldots, n\}$ be an orthonormal basis of $T_{\sigma(a)}M$ such that

$$e_1 = \dot{\sigma}(a).$$

Further, let E_1 be the unit tangent field $\dot{\sigma}$ and $E_2, \ldots, E_n$ normal vector fields along σ parallel with respect to the normal connection $\nabla^\perp$ of the normal bundle $\dot{\sigma}^\perp$ such that

$$E_i(a) = e_i, \quad i = 2, \ldots, n.$$

Then, the Fermi coordinates $(x^1, \ldots, x^n)$ with respect to $\sigma(a)$ and $(E_1, \ldots, E_n)$ are defined by

$$x^1(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j) = t - a,$$

$$x^i(\exp_{\sigma(t)} \sum_{j=2}^{n} t^j E_j) = t^i, \quad \text{where } 2 \leq i \leq n.$$

For a vector $v \in T^\perp_{\sigma(t)}\sigma$ with $\exp_{\sigma(t)} v \in U$ we have

$$v = \sum_{a=2}^{n} x^a E_a(t) = ru$$

where $\| u \| = 1$ and $r^2 = \sum_{a=2}^{n} (x^a)^2$.

We put

$$\varkappa_u = g(\ddot{\sigma}, u) \tag{3}$$

where $\ddot{\sigma}(t)$ is the (mean) curvature normal of σ at $\sigma(t)$. Note that, when u is parallel with respect to $\nabla^\perp$, we have

$$\nabla_{\dot{\sigma}} u = -\varkappa_u \dot{\sigma}.$$

Finally we note that the local reflection φ_σ is now given by

$$\varphi_\sigma : (x^1, x^2, \ldots, x^n) \mapsto (x^1, x^2, \ldots, x^n).$$

To compute the tension field for φ_σ we need the Christoffel symbols with respect to the system of Fermi coordinates. Therefore we use

$$m\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left\{ \frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} \frac{\partial g_{ij}}{\partial x^\ell} \right\}. \tag{4}$$

So we have to compute g_{ij}, g^{ij}. Therefore we use the method developed in [7] or use Jacobi vector fields as in [6], [9]. The result is given in the following

LEMMA 3. *Let* $m = \sigma(t)$ *and* $p = \exp_{\sigma(t)}(su)$, $\|u\| = 1$. *With respect to the Fermi coordinates* $(x^1, \ldots, x^n)$ *introduced before, we have*

$$g_{11}(p) = \sum_{\ell=0}^{4} \alpha_{11}^{\ell}(m)\, s^{\ell} + 0(s^5) \tag{5}$$

where

$$\alpha_{11}^{0} = 1,$$

$$\alpha_{11}^{1} = -2\varkappa_u,$$

$$\alpha_{11}^{2} = \varkappa_u^2 - R_{1u1u},$$

$$\alpha_{11}^{3} = -\frac{1}{3}(\nabla_u R_{1u1u} - 4\varkappa_u R_{1u1u}),$$

$$\alpha_{11}^{4} = -\frac{1}{12}\left(\nabla R_{uu}^2 R_{1u1u}\, 4R_{1u1u}^2 - 4\sum_{c=2}^{n} R_{1ucu}^2 - 6\varkappa_u \nabla_u R_{1u1u} + 4\varkappa_u^2 R_{1u1u}\right);$$

$$g_{1a}(p) = \sum_{\ell=0}^{4} \alpha_{1a}^{\ell}(m)\, s^{\ell} + 0(s^5), \quad a = 2, \ldots, n, \tag{6}$$

where

$$\alpha_{1a}^{0} = \alpha_{1a}^{1} = 0,$$

$$\alpha_{1a}^{2} = -\frac{2}{3} R_{1uau},$$

$$\alpha_{1a}^{3} = -\frac{1}{12}(3\nabla_u R_{1uau} - 4\varkappa_u R_{1uau}),$$

$$\alpha_{1a}^{4} = -\frac{1}{30}\left(2\partial_{uu}^2 R_{1uau} - 4R_{1u1u}R_{1uau}\, 4\sum_{c=2}^{n} R_{1ucu}R_{aucu} - 5\varkappa_u \nabla_u R_{1uau}\right);$$

$$g_{ab}(p) = \sum_{\ell=0}^{4} \alpha_{ab}^{\ell}(m)\, s^{\ell} + 0(s^5), \quad a,b = 2, \ldots, n, \tag{7}$$

where

$$\alpha_{ab}^{0} = \delta_{ab},$$

$$\alpha_{ab}^{1} = 0,$$

$$\alpha_{ab}^{2} = -\frac{1}{3} R_{uaub},$$

$$\alpha_{ab}^{3} = -\frac{1}{6}\nabla_u R_{uaub},$$

$$\alpha_{ab}^{4} = -\frac{1}{180}\left(9\nabla_{uu}^2 R_{uaub} - 8R_{1uau}R_{1ubu} - 8\sum_{c=2}^{n} R_{uauc}^2 R_{ubuc}\right);$$

(8) $$g^{11}(p) = \sum_{\ell=0}^{4} \beta_{11}^{\ell}(m)\, s^{\ell} + O(s^5),$$

where

$$\beta_{11}^{0} = 1,$$

$$\beta_{11}^{1} = 2\varkappa_u,$$

$$\beta_{11}^{2} = R_{1u1u} + 3\varkappa_u^2,$$

$$\beta_{11}^{3} = \frac{1}{3}(\nabla_u R_{1u1u} + 8\varkappa_u R_{1u1u} + 12\varkappa_u^3),$$

$$\beta_{11}^{4} = \frac{1}{36}\left(3\nabla_{uu}^2 R_{1u1u} + 24 R_{1u1u}^2 + 4\sum_{c=2}^{n} R_{1ucu}^2 + 30\varkappa_u \nabla_u R_{1u1u} + 180\varkappa_u^2 R_{1u1u} + 180\varkappa_u^4\right);$$

(9) $$g^{1a}(p) = \sum_{\ell=0}^{4} \beta_{1a}^{\ell}(m)\, s^{\ell} + O(s^5), \quad a = 2, \ldots, n,$$

where

$$\beta_{1a}^{0} = \beta_{1b}^{1} = 0,$$

$$\beta_{1a}^{2} = \frac{2}{3} R_{1uau},$$

$$\beta_{1a}^{3} = \frac{1}{4}(\nabla_u R_{1uau} + 4\varkappa_u R_{1uau}),$$

$$\beta_{1a}^{4} = \frac{1}{45}\left(3\nabla_{uu}^2 R_{1uau} + 24 R_{1u1u} R_{1uau} + 4\sum_{c=2}^{n} R_{1ucu} R_{aucu} + 15\varkappa_u \nabla_u R_{1uau} + 60\varkappa_u^2 R_{1uau}\right);$$

(10) $$g^{ab}(p) = \sum_{\ell=0}^{4} \beta_{ab}^{\ell}(m)\, s^{\ell} + O(s^5), \quad a,b = 2, \ldots, n,$$

where

$$\beta_{ab}^{0} = \delta_{ab},$$

$$\beta_{ab}^{1} = 0,$$

$$\beta_{ab}^{2} = \frac{1}{3} R_{uaub},$$

$$\beta_{ab}^{3} = \frac{1}{3} \nabla_u R_{uaub},$$

$$\beta_{ab}^{4} = \frac{1}{60}\left(3\nabla_{uu}^2 R_{uaub} + 24 R_{1uau} R_{1ubu} + 4\sum_{c=2}^{n} R_{uauc} R_{ubuc}\right).$$

4. Harmonic reflections

It follows from (2) and the local expression for the reflection φ_σ (section 3) that φ_σ is harmonic if and only if

$$\tau(\varphi_\sigma)^1(p) = \{g^{11}\nabla(d\varphi_\sigma)^1_{11} + 2g^{1a}\nabla(d\varphi_\sigma)^1_{1a} + g^{ab}\nabla(d\varphi_\sigma)^1_{ab}\}(p) = 0, \tag{11}$$

$$\tau(\varphi_\sigma)^c(p) = \{g^{11}\nabla(d\varphi_\sigma)^c_{11} + 2g^{1a}\nabla(d\varphi_\sigma)^c_{1a} + g^{ab}\nabla(d\varphi_\sigma)^c_{ab}\}(p) = 0, \tag{12}$$

for a, b, c, = 2, ..., n, where

$$\nabla(d\varphi_\sigma)^1_{11}(p) = -\Gamma^1_{11}(p) + \Gamma^1_{11}(\varphi_\sigma(p)),$$

$$\nabla(d\varphi_\sigma)^1_{1a}(p) = -\Gamma^1_{1a}(p)\,\Gamma^1_{1a}(\varphi_\sigma(p)),$$

$$\nabla(d\varphi_\sigma)^1_{ab}(p) = -\Gamma^1_{ab}(p) + \Gamma^1_{ab}(\varphi_\sigma(p)),$$

$$\nabla(d\varphi_\sigma)^c_{11}(p) = \Gamma^c_{11}(p) + \Gamma^c_{11}(\varphi_\sigma(p)),$$

$$\nabla(d\varphi_\sigma)^c_{1a}(p) = \Gamma^c_{1a}(p)\,\Gamma^c_{1a}(\varphi_\sigma(p)),$$

$$\nabla(d\varphi_\sigma)^c_{ab}(p) = \Gamma^c_{ab}(p) + \Gamma^c_{ab}(\varphi_\sigma(p)).$$

It is worthwile to note here that $\nabla(d\varphi_\sigma)^1_{1a}$, $\nabla(d\varphi_\sigma)^c_{11}$, $\nabla(d\varphi_\sigma)^c_{ab}$ are even functions and $\nabla(d\varphi_\sigma)^1_{11}$, $\nabla(d\varphi_\sigma)^1_{ab}$, $\nabla(d\varphi_\sigma)^c_{1a}$ are odd functions.

In order to write down the explicit expressions we shall need, we put

$$\frac{\partial g_{ij}}{\partial x^k}(p) = \sum_{\ell=0}^{4} \gamma^\ell_{ijk}(m)\, s^\ell + O(s^5)$$

and

$$\nabla(d\varphi_\sigma)^k_{ij}(p) = \sum_{\ell=0}^{4} A^\ell_{ijk}(m)\, s^\ell + O(s^5)$$

for i, j, k = 1, ..., n.

A lengthy computation, which we omit, then leads to

$$\begin{aligned}\tau(\varphi_\sigma)^c(p) = &-\gamma^0_{11c}(m) - \beta^1_{11}(m)\gamma^0_{11c}(m)s + \left(A^2_{11c} + \sum_{a=2}^{n} A^2_{aac} - \gamma^0_{11c}\beta^2_{11}\right)(m)s^2 + \\ &+ \left(\beta^3_{11}\gamma^0_{11c} + 2\sum_{a=2}^{n}\beta^2_{1a}(\gamma^1_{1ca} - \gamma^1_{1ac}) + \beta^1_{11}A^2_{11c}\right)(m)s^3 + O(s^4)\end{aligned} \tag{13}$$

where

$$\gamma^0_{11c}(m) = -2g(\ddot\sigma, E_c)(t). \tag{14}$$

Now we are ready to give the

Proof of Theorem 1 – For a harmonic reflection φ_σ we have $\tau(\varphi_\sigma)^c = 0$, c = 2, ..., n, for all $p \in U$. This implies from (13)

$$\gamma^0_{11c}(m) = \gamma^0_{11c}(\sigma(t)) = 0$$

for all $t \in [a, b]$. Hence, since $\ddot{\sigma}$ is orthogonal to $\dot{\sigma}$, we get from (14) that $\ddot{\sigma} = 0$ which proves the required result.

For the rest of this section we suppose that s is a geodesic. Hence $\varkappa_u = 0$ for all u and so

$$v^0_{11c} = 0, \quad c = 2, \ldots, n. \tag{15}$$

Also, from (8), we have

$$\beta^1_{11} = 0. \tag{16}$$

To prove Theorem 2 we shall use the following

LEMMA 4 - [1], [2]. *Let* (M, g) *be a connected* m*-dimensional Riemannian manifold with* $m > 2$. *Then* (M, g) *is space of constant curvature if and only if*

$$R(X, Y, X, Z) = 0$$

for any orthogonal triple of vector fields X, Y, Z, *on* M.

Now we give the

PROOF OF THEOREM 2 - First, let (M, g) be a space of constant curvature. Then, it is proved in [9] it that the local reflections φ_σ with respect to all geodesics σ are isometries. So, since any isometry is a harmonic map, all φ_σ are harmonic.

To prove the converse, we start with $\tau(\varphi_\sigma)^c = 0$, $c = 2, \ldots, n$. Using (15) and (16), (13) implies

$$\sum_{a=2}^{n} \beta^2_{1a}(\gamma^1_{1ca} - \gamma^1_{1ac}) = 0. \tag{17}$$

From (9) we have

$$\beta^2_{1a} = \frac{2}{3} R_{1uau}$$

and from the detailed computations, we obtain

$$\gamma^1_{1ca} - \gamma^1_{1ac} = -\frac{2}{3}(R_{1cua}\, R_{1auc} + 2R_{1uca}).$$

Hence (17) leads to

$$\sum_{a=2}^{n} R_{1uau}(R_{1cua} - R_{1auc} + 2R_{1uca}) = 0. \tag{18}$$

Now, since u is an arbitrary unit vector normal to the geodesic σ, we may put $u = E_c$. Then (18) implies

$$\sum_{a=2}^{n} R^2_{1uau} = 0.$$

Since this must be valid for all geodesics, the required result follows at once from Lemma 4 when $\dim M > 2$.

So we are left with the two-dimensional case. Here, the detailed computations show easily that, with $u = E_2$, we have

$$g_{11}(p) = 1 - s^2 R_{1u1u}(m) - \frac{1}{3} s^3 \nabla_u R_{1u1u}(m) = O(s^4),$$
$$g_{12}(p) = O(s^5),$$
$$g_{22}(p) = 1 + O(s^5),$$

and

$$g^{11}(p) = 1 + s^2 R_{1u1u}(m) + \frac{1}{3} s^3 \nabla_u R_{1u1u}(m) + O(s^4),$$
$$g^{12}(p) = O(s^5),$$
$$g^{22}(p) = 1 + O(s^5).$$

A similar calculation as before now gives

$$\tau(\varphi_\sigma)^2(p) = - s^2 (\nabla_u R_{1u1u})(m) + O(s^3)$$

and hence, when φ_σ is harmonic, we must have

$$\nabla_u R_{1u1u} = 0$$

at any point of (M, g) and for all geodesics σ. This implies that the two-dimensional connected manifold has consant curvature.

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