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# Some results for an optimal control problem with a semilinear state equation

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**Analisi matematica.** – *Some results for an optimal control problem with a semilinear state equation.* Nota di FAUSTO GOZZI, presentata (*) dal Corrisp. R. CONTI.

ABSTRACT. – We consider a quadratic control problem with a semilinear state equation depending on a small parameter $\epsilon$. We show that the optimal control is a regular function of such parameter.

KEY WORDS: Optimal control; Semilinear state equation; Hamilton-Jacobi equation.

RIASSUNTO. – *Un risultato per un problema di controllo ottimale con equazione di stato semilineare.* Si considera un problema di controllo quadratico con una equazione di stato semilineare dipendente da un piccolo parametro $\epsilon$, e si prova che il controllo ottimale è una funzione regolare di tale parametro.

## 1. INTRODUCTION

We consider a dynamical system governed by the following semilinear state equation:

$$[1.1] \quad \begin{cases} y' = Ay + \epsilon f(y) + Bu & \text{on } [0, T]; \\ y(0) = x & x \in H; \end{cases}$$

where $A : D(A) \subset H \to H$ and $B : U \to H$ are linear operators in the Hilbert spaces H and U respectively, and f is a regular function from H into H.

We consider then the following optimal control problem:

$$[P] \quad \begin{cases} \text{Minimize the functional:} \\ [1.2] \quad J(u) = \frac{1}{2}\int_0^T (\langle My, y\rangle + \langle Nu, u\rangle)\, ds + \frac{1}{2}\langle P_0 y(T), y(T)\rangle \\ \text{over all controls } u \in L^2(0, T; U)), \\ \text{where } y \text{ is subject to the state equation [1.1];} \end{cases}$$

M, N, $P_0$, are linear operators which we will define in the next section. If $\epsilon = 0$ then the problem [P] reduces to the well known linear quadratic problem which has been extensively studied (see for instance [6]).

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We shall prove that, if the parameter $\varepsilon$ is sufficiently small and the data are sufficiently regular, our problem admits at least one optimal control $u_\varepsilon$ which is continuous as a function of the parameter $\epsilon$ at $\epsilon = 0$. Moreover the value function $\psi_\epsilon$ of the problem is lipschitz continuous in $\epsilon$ in a neighbourhood of 0.

## Notations and statement of the main result

If H is a Hilbert space we shall denote by $\mathcal{L}(H)$ the Banach algebra of the linear bounded operators from H into H. By $\Sigma(H)$ we represent the set of all hermitian operators in $\mathcal{L}(H)$ and we set:

$$\Sigma^+ = \{T \in \mathcal{L}(H); \quad (Tx, x) \geq 0 \quad \forall x \in H\}$$

If U is another Hilbert space we denote by $\mathcal{L}(U, H)$ the set of all linear bounded operators from U into H. Finally we say that $f \in C^1_{Lip}(H, H)$ if $f: H \to H$ is a differentiable function and $f, f'$ are lipschitz continuous and bounded on bounded sets of H.

In the following we work in two Hilbert spaces:

H state space and U control space

We are concerned with the control problem [P] under the following assumptions:

a) A is the infinitesimal generator of an analytic semigroup $e^{tA}$ in H;
b) $e^{tA}$ is compact for any $t > 0$;
c) $B \in \mathcal{L}(U, H)$;
d) $M, P_0 \in \Sigma^+(H)$;
e) $N \in \Sigma^+(U)$, $N \geq \alpha I$ for some $\alpha > 0$;
f) $f \in C^1_{Lip}(H, H)$;
g) $\epsilon \in [-\epsilon_0, \epsilon_0]$ for some fixed $\epsilon_0 > 0$.

We first remark (see for instance [5]) that if $\epsilon_0$ is sufficiently small, the state equation [1.1] has a unique mild solution on [0, T], that is there exists $y \in C([0,T]; H)$ which satisfies [1.1] in integral form: (see again [5])

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A}[Bu(s) + \epsilon f(y(s))]\,ds \tag{2.2}$$

Since $\epsilon_0$ depends on $|x|_H$ and $|u|_{L^2(0,T;U)}$ we have to work with x and u belonging to some ball of H and $L^2(0, T; U)$ respectively. In particular we must minimize the functional J on a ball $B_r$ of $L^2(0, T; U)$. However, if r is sufficiently large, this is equivalent to minimize J on all space $L^2(0,T;U)$. So, in the following we limit ourself to study this case.

Moreover the assumption [2.1] – b) implies that the map $\vartheta: L^2(0,T;U) \to C([0,T];H)$, $u \to y$, is compact by the Ascoli theorem. This gives, by standard arguments (see [3]) that the problem [P] admits at least one solution $(u_\epsilon^*, y_\epsilon^*) \in L^2(0,T;U) \times C([0,T];H)$.

The value function of the problem is given by:

[2.3]
$$\psi_\epsilon(\tau, x) = \inf\left\{\frac{1}{2}\int_t^T (\langle My, y\rangle + \langle Nu, u\rangle) ds + \frac{1}{2}\langle\langle P_0 y(T), y(T)\rangle\rangle; \right.$$
$$u \in L^2(t,T;U),\ y \text{ solution of:}$$
$$[2.4]\left\{\begin{array}{l} y' = Ay + \epsilon f(y) + Bu \\ y(t) = x \end{array}\right\}$$
$$\overset{\text{def}}{=} \inf_{u \in L^2(t,T;U)} J_\epsilon(t,x,u) = J_\epsilon(t,x,u_\epsilon^*)$$

where the last equality follows from Bellman's optimality principle.

The Hamilton-Jacobi equation associated with the control problem [P] is (setting $K = BN^{-1}B^*$):

[2.5]
$$\left\{\begin{array}{l} \psi_t - \frac{1}{2}\langle K\psi_x, \psi_x\rangle + \langle Ax + \epsilon f(x), \psi_x\rangle + \frac{1}{2}\langle Mx, x\rangle = 0 \quad \forall (t,x) \in [0,T] \times D(A) \\ \\ \psi(T,x) = \frac{1}{2}\langle P_0 x, x\rangle \qquad \forall x \in H \end{array}\right.$$

The following theorem concerning the function $\psi_\epsilon$ is proved in [1]:

THEOREM (2.1) – *Under the assumptions* [2.1] *the value function* $\psi_\epsilon$ *of the problem* [P] *satisfies the following properties*:

*A)* $\psi_\epsilon: [0,T] \times H \to \mathbb{R}$ *is continuous.*

*B)* $\psi_\epsilon(t, \cdot)$ *is lipschitz continuous on bounded sets of* H.

*C)* $\psi_\epsilon(\cdot, x)$ *is absolutely continuous* $\forall x \in D(A)$.

*D)* $D_x^-\psi_\epsilon(t,x) \neq 0 \quad \forall (t,x) \in [0,T] \times H$,
*where* $D_x^-$ *denotes the subdifferential with respect to the variable* x.

*E) For every* $x \in D(A)$ *there exists* $\eta \in D_x^-\psi_\epsilon(t,x)$ *such that:*

$$\psi_{\epsilon t} - \frac{1}{2}\langle K\eta, \eta\rangle + \langle Ax - \epsilon f(x), \eta\rangle - \frac{1}{2}\langle Mx, x\rangle = 0 \qquad on\ [0,T],$$

*and*

$$\psi_\epsilon(T, x) = \frac{1}{2}\langle P_0 x\rangle \qquad \forall x \in H$$

*Moreover* $\psi_\epsilon$ *is a viscosity solution of the Hamilton-Jacobi equation* [2.5] *(see* [4]*).*

*F) Any optimal control* $u_\epsilon^*$ *can be expressed as a function of the corresponding optimal state* $y_\epsilon^*$ *with the feedback law*:

$$u_\epsilon^*(t) = -N^{-1}B^*\eta$$

$$\forall t \in [0,T] \text{ and for some } \eta \in D_x^- \psi_\epsilon(t, y_\epsilon^*(t))$$

The main result of this paper is the following:

THEOREM (2.2) - *The following statements hold:*

*I) The value function of the problem* [P], $\psi_\epsilon(t,x)$, *is lipschitz continuous with respect to* $\epsilon \in [-\epsilon_0, \epsilon_0]$, *uniformly in* $(t,x)$ *on bounded sets of* $[0,T] \times H$.

*II) If* $u_\epsilon^*$ *denotes any fixed optimal control, then there exists the limit*:

$$\lim_{t \to 0} u_\epsilon^* = u_0^* \qquad in\ C[0,T]; U)$$

## 3. PROOF OF THE THEOREM (2.2)

We need two preliminary results:

LEMMA (3.1) - (PONTRYAGIN MAXIMUM PRINCIPLE) *If the pair* $(u_\epsilon^*, y_\epsilon^*)$ *is optimal for the problem* [P], *then there exists* $p_\epsilon \in ([0,T]; H)$ *such that*:

$$\text{[3.1]} \quad \begin{cases} p_\epsilon' + (A + \epsilon f'(y_\epsilon^*))^* p_\epsilon = -My_\epsilon^*; & p_\epsilon(T) = P_0 y_\epsilon^*(T) \\ u_\epsilon^* = -N^{-1}B_* p_\epsilon; & \\ (y_\epsilon^*)' = Ay_\epsilon^* + Bu_\epsilon^* + \epsilon f(y_\epsilon^*); & y(0) = x \end{cases}$$

These equations are called the optimality conditions for the problem [P]. The proof is standard (see for example [3])

LEMMA (3.2) - (REGULARITY OF OPTIMAL CONTROL) *Let* $(u_\epsilon^*, y_\epsilon^*)$ *be any optimal pair in problem* [P]. *Then there exists* $\alpha \in (0,1)$ *such that*:

$$\text{[3.2]} \qquad u_\epsilon^* \in C([0,T]; U) \cap C^\alpha([\beta, T-\beta]; U) \quad \forall \beta \in \left(0, \frac{T}{2}\right);$$

*and*

$$\text{[3.3]} \qquad y_\epsilon^* \in C([0,T); U) \cap C^\alpha([\beta]; H) \cap C^{1,\alpha}([\beta, T-\beta]; H) \quad \forall \beta \in \left(0, \frac{T}{2}\right);$$

*Proof.* - We have $u_\epsilon^* \in L^2(0,T;U)$ and this implies that $y_\epsilon^* \in C^\alpha([\beta,T];H)$ for some $\alpha \in (0,1)$ (see [7] p. 110), and therefore $p_\epsilon \in C^\alpha([\beta, T-\beta]; H)$ (see again [7] p. 168).

Hence $u^*_\epsilon$ is Hölder continuous on $[\beta, T-\beta]$, and $y^*_\epsilon$ is a classical solution of [1.1]. From [7] p. 115 our statement follows.

Q.E.D

Now we can prove Theorem (2.2).

I) We write, for convenience:

$$\psi_\epsilon = \psi(t,x,\epsilon) = \inf_{u \in L^2(t,T;U)} J_\epsilon(t,x,u) = J_\epsilon(t,x,u^*_\epsilon); \tag{3.4}$$

Let $\epsilon_1, \epsilon_2 \in [-\epsilon_0, \epsilon_0]$; we have:

$$\psi(t,x,\epsilon_1) - \psi(t,x,\epsilon_2) \le J_{\epsilon_1}(t,x,u^*_{\epsilon_2}) - J_{\epsilon_1}(t,x,u^*_{\epsilon_2}) = \tag{3.5}$$

$$= \frac{1}{2}\int_t^T (\langle M\tilde{y}_{\epsilon_2}, \tilde{y}_{\epsilon_2}\rangle - \langle My^*_{\epsilon_2}, y^*_{\epsilon_2}\rangle)\, ds + \frac{1}{2}\langle P_0\tilde{y}_{\epsilon_2}(T), \tilde{y}_{\epsilon_2}(T)\rangle - \frac{1}{2}\langle P_0 y^*_{\epsilon_2}(T), y^*_{\epsilon_2}(T)\rangle$$

where $\tilde{y}_{\epsilon_2}$ is the "mild" solution of the Cauchy problem:

$$\begin{cases} y' = Ay + \epsilon_1 f(y) + Bu^*_{\epsilon_2} \\ y(t) = x \end{cases} \tag{3.6}$$

which is equivalent to:

$$\tilde{y}_{\epsilon_2}(s) = e^{(s-t)A}x + \int_t^s e^{(s-\sigma)A}\epsilon_2 f(\tilde{y}_{\epsilon_2}(\sigma))\, d\sigma + \int_t^s e^{(s-\sigma)A} Bu^*_{\epsilon_2}(\sigma)\, ds; \tag{3.7}$$

and $y^*_{\epsilon_2}$ is the optimal state given implicitely by the formula:

$$y^*_{\epsilon_2}(s) = e^{(s-t)A}x + \int_t^s e^{(s-\sigma)A}\epsilon_2 f(y^*_{\epsilon_2}(\sigma))\, d\sigma + \int_t^s e^{(s-\sigma)A} Bu^*_{\epsilon_2}(\sigma)\, ds; \tag{3.8}$$

We remark that a unique "mild" solution $\tilde{y}_{\epsilon_2}$ of [3.7] does exists (see [5]) and the following estimates hold (by the contractions principle):

$$\begin{cases} |y^*_{\epsilon_2}(s)|_H \le C_0(|x|_H + |u^*_{\epsilon_2}|_{L^2(t,T;U)}) \\ |\tilde{y}_{\epsilon_2}(s)|_H \le C_0(|x|_H + |u^*_{\epsilon_2}|_{L^2(0,T;U)}) \end{cases} \tag{3.9}$$

with $C_0$ independent of $\epsilon_2$.

Moreover we have, if $|x| \le r_0$ ($r_0$ fixed):

$$\int_0^T |u^*_{\epsilon_2}(s)|^2_U\, ds \le \frac{1}{\alpha} J_{\epsilon_2}(t,x,u_{\epsilon_2}) \le \frac{1}{\alpha} J_{\epsilon_2}(t,x,0) \le C_{r_0}; \tag{3.10}$$

where $C_{r_0}$ depends only on $|x| \le r_0$:

It follows, if

$$|\langle M\tilde{y}_{\epsilon_2}(s), \tilde{y}_{\epsilon_2}(s)\rangle - \langle My^*_{\epsilon_2}(s), y^*_{\epsilon_2}(s)\rangle| \le C_1 |\tilde{y}_{\epsilon_2}(s) - y^*_{\epsilon_2}(s)|_H. \tag{3.11}$$

By the Gronwall lemma we obtain:

$$[3.12] \qquad |\tilde{y}_{\epsilon_2}(s) - y^*_{\epsilon_2}(s)|_H \leq C_2 |\epsilon_1 - \epsilon_2| ,$$

with $C_2$ independent of s and $\epsilon$.

Now if we return to inequality [3.5] we have:

$$\psi(t, x, \epsilon_1) - \psi(t, x, \epsilon_2) \leq L_0 |\epsilon_1 - \epsilon_2|$$

where $L_0$ depends only on $r_0$.

Analogously we have:

$$\psi(t, x, \epsilon_2) \leq \psi(t, x, \epsilon_1) \leq L_1 |\epsilon_1 - \epsilon_2|$$

with $L_1$ depending only on $r_0$ and this completes the proof of I).

II) By [3.10] we have for $|x|_H \leq r$:

$$|u^*_\epsilon|_{L^2(0,T;U)} \leq C_r$$

where $C_r$ is a constant independent of $\epsilon$.

This implies that $\{u_\epsilon\}$ is weakly compact in $L^2(0, T; U)$; therefore on a subsequence, still denoted by $\{u_\epsilon\}$ we have, for $\epsilon \to 0$ and any $\bar{u} \in L^2(0, T; U)$:

$$u_\epsilon \xrightarrow{\epsilon \to 0} \bar{u} \quad \text{weakly in } L^2(0, T; U)$$

but from compactness of $e^{tA}$ it follows:

$$\left.\begin{array}{l} y_\epsilon \xrightarrow{\epsilon \to 0} \bar{y} \\ p_\epsilon \xrightarrow{\epsilon \to 0} \bar{p} \\ u^*_\epsilon \xrightarrow{\epsilon \to 0} \bar{u} \end{array}\right\} \text{ strongly in } C([0, T]; H)$$

where $\bar{y}$, $\bar{p}$ satisfy the equations of the maximum principle:

$$\begin{cases} y' = Ay + Bu & y(0) = x; \\ p' = -A^*p - My & p(T) = P_0 y(T); \\ u = -N^{-1}B^*p \end{cases}$$

this implies that $\bar{u} = u_0, \bar{y} = y_0$ since the optimal control is unique in the case $\epsilon = 0$.

Finally, by contradiction, if $|u^*_{\epsilon_n} - u_0|_{C([0,T];U)} \geq h > 0$ for some subsequence, we show with the same arguments, that $u^*_{\epsilon_n}$ has a limit point $\hat{u}$ and $\hat{u} = u_0$.

Q.E.D.

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