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A minimum principle in the dynamics of elastic materials with voids

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Fisica matematica. — *A minimum principle in the dynamics of elastic materials with voids* (*). Nota (**) di MICHELE CIARLETTA e EDOARDO SCARPETTA (***), presentata dal Socio T. MANACORDA.

ABSTRACT. — In the context of the linear, dynamic problem for elastic bodies with voids, a minimum principle in terms of mechanical energy is stated. Involving a suitable (Reiss type) function in the minimizing functional, the minimum character achieved in the Laplace-transform domain is preserved when going back to the original time domain. Initial-boundary conditions of quite general type are considered.

KEY WORDS: Variational; Minimum; Principle.

RIASSUNTO. — *Un principio di minimo nella dinamica dei materiali elastici con vuoti.* Nell'ambito dell'elastodinamica lineare per solidi porosi, si dimostra un principio di minimo in termini dell'energia meccanica. Introducendo un'opportuna funzione (tipo Reiss) nel funzionale in oggetto, il carattere di minimo ottenuto nel dominio delle trasformate di Laplace viene conservato tornando al dominio temporale originario. Si considerano inoltre condizioni iniziali ed al contorno alquanto generali.

1. INTRODUCTION.

The aim of this paper is to establish a minimum principle for the linear dynamic problem of *elastic* bodies having small distributed *voids* in their constituent material. The voids are assumed not to contain anything of mechanical or energetic significance.

The exact theory of such *porous* solids is originally due to Nunziato and Cowin [1], who later also gave the linear version of it [2] (1); this theory intends to represent the mechanical behaviour of various kinds of geological materials, such as rock and soils, or porous manufactured materials, for which the classical continuum model appears to be unsatisfactory. We refer to [1] (cf. also [6]) for more details and physical insights into the basic concepts underlying the theory.

The mixed initial-boundary-value problem of a linear elastic body with voids has been set up in [2] and [7], where, among other things, existence and uniqueness of regular solutions is dealt with. In the latter paper, a minimum potential energy principle is stated for the equilibrium problem, whereas only variational theorems are provided for the dynamic one.

In this note, we thus prove a minimum principle for the dynamic problem in the original time field, allowing non-homogeneous initial conditions; moreover, surface stress boundary conditions of dissipative type are also taken into account. We follow the guidelines of Reiss' procedure [8] for seeking minimal functionals in classical elastodynamics, here extended to the more general theory and boundary conditions at

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(1) Extensions to thermodynamic context have been performed in [3, 4] and [5] for the non-linear and linear case, respectively.

hand. In essence, this procedure moves from minimum principles established in the Laplace-transform domain (cf. [9]), to then go back to the original time domain without losing the minimum character.

The claimed minimum principle is formulated in section 3, after having stated the equations governing the problem in section 2.

2. Relevant equations and definitions.

According to [2, 7], the linear dynamic problem for an elastic body $\mathcal{B}$ with voids is governed by the following local balances of momentum and equilibrated force:

$$\mathrm{Div}\,\boldsymbol{T} + \rho\boldsymbol{b} = \rho\ddot{\boldsymbol{u}} \qquad \mathrm{Div}\,\boldsymbol{h} + g + \rho l = \rho k\ddot{\varphi}\,, \qquad \text{in } \Omega_T \equiv \Omega \times (0, T)\,. \tag{1}$$

In these equations, Ω is the bounded, smooth domain of R^3 occupied by $\mathcal{B}$ in some fixed reference configuration $\boldsymbol{k}$ and T a positive number ($\leqslant +\infty$); $\boldsymbol{T}$ is the symmetric stress tensor and $\boldsymbol{h}$ the equilibrated stress vector. Moreover, g is the intrinsic equilibrated body force, $\boldsymbol{b}$ the external body force, l the extrinsic equilibrated body force, ρ the *bulk* mass density and k the equilibrated inertia. Finally, $\boldsymbol{u}$ and φ are the independent kinematical variables of the theory: the displacement vector field from $\boldsymbol{k}$ and the change in volume fraction with respect to $\boldsymbol{k}$ (2), respectively [1, 2]. Such a pair $(\boldsymbol{u}, \varphi)$ represents a *motion* for $\mathcal{B}$.

To (1) we append the appropriate constitutive equations, which read as follows [2]:

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}(\boldsymbol{u}, \varphi) = \boldsymbol{C}\boldsymbol{E} + \boldsymbol{D}\,\nabla\varphi + \boldsymbol{B}\varphi \\ \boldsymbol{h} &= \boldsymbol{h}(\boldsymbol{u}, \varphi) = \boldsymbol{A}\,\nabla\varphi + \boldsymbol{D}\boldsymbol{E} + \boldsymbol{f}\varphi \\ g &= g(\boldsymbol{u}, \varphi, \dot{\varphi}) = -\,\omega\dot{\varphi} - \xi\varphi - \boldsymbol{B}:\boldsymbol{E} - \boldsymbol{f}\cdot\nabla\varphi \end{aligned} \qquad \text{in } \Omega_T\,. \tag{2}$$

In (2), where $\boldsymbol{E} = 1/2[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t]$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are fourth-order and third-order tensors, respectively, and $\boldsymbol{A}$, $\boldsymbol{B}$ second-order tensors (in component form: $(\boldsymbol{CE})_{ij} = C_{ijkl}E_{kl}$, $(\boldsymbol{D}\,\nabla\varphi)_{ij} = D_{ijk}(\nabla\varphi)_k$, $(\boldsymbol{DE})_i = D_{ijk}E_{jk}$, $(\boldsymbol{A}\,\nabla\varphi)_i = A_{ij}(\nabla\varphi)_j$, $\boldsymbol{B}:\boldsymbol{E} = B_{ij}E_{ij}$, $\boldsymbol{f}\cdot\nabla\varphi = f_i(\nabla\varphi)_i$, etc.; $i, j, k, l = 1, 2, 3$). Together with the vector $\boldsymbol{f}$ and the scalars ω and ξ, the above tensor fields charaterize pointwise the material properties of the body; their components obey the following symmetry relations in Ω [2]:

$$C_{ijkl} = C_{klij} = C_{jikl}\,; \qquad D_{ijk} = D_{ikj} = D_{jik}\,; \qquad A_{ij} = A_{ji}\,; \qquad B_{ij} = B_{ji}\,.$$

In the sequel, we shall assume that the quadratic form

$$\frac{1}{2}\boldsymbol{C}\boldsymbol{M}:\boldsymbol{M} + \frac{1}{2}\boldsymbol{A}\boldsymbol{m}\cdot\boldsymbol{m} + \frac{1}{2}\xi\mu^2 + \mu\boldsymbol{B}:\boldsymbol{M} + \boldsymbol{D}\boldsymbol{M}\cdot\boldsymbol{m} + \mu\boldsymbol{f}\cdot\boldsymbol{m}$$

is *positive definite* for each (second-order) tensor $\boldsymbol{M}$, vector $\boldsymbol{m}$ and scalar μ (cf. [2, sect. 2], [7, sect. 3]). Note that for $\boldsymbol{M} \equiv \boldsymbol{E}$, $\boldsymbol{m} \equiv \nabla\varphi$, $\mu \equiv \varphi$, the above expression represents the *potential energy* density of $\mathcal{B}$ associated to the motion $(\boldsymbol{u}, \varphi)$. Moreover, it must be also $\omega \geqslant 0$, $k \geqslant 0$ according to a thermodynamic argument [1].

The formulation of the present initial-boundary-value problem is completed on setting:

$$\boldsymbol{u} = \boldsymbol{u}_0\,, \quad \dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_0\,; \qquad \varphi = \varphi_0\,, \quad \dot{\varphi} = \dot{\varphi}_0 \qquad \text{in } \Omega \times \{0\}\,, \tag{3}$$

(2) In the reference configuration $\boldsymbol{k}$ the volume fraction field is assumed constant.

and

$$(4)\qquad \begin{aligned} &\boldsymbol{u} = \boldsymbol{u}_\Sigma \quad \text{in } \partial_1\Omega \times [0,T)\,, && \varphi = \varphi_\Sigma \quad \text{in } \partial_3\Omega \times [0,T)\,,\\ &\boldsymbol{Tn} = \boldsymbol{t}_\Sigma - \boldsymbol{\Gamma}\dot{\boldsymbol{u}} \quad \text{in } \partial_2\Omega \times [0,T)\,, && \boldsymbol{h}\cdot\boldsymbol{n} = h_\Sigma - \gamma\dot{\varphi} \quad \text{in } \partial_4\Omega \times [0,T)\,, \end{aligned}$$

where $\boldsymbol{n}$ is the outward unit normal, $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0$, φ_0, $\dot{\varphi}_0$, $\boldsymbol{u}_\Sigma$, φ_Σ, $\boldsymbol{t}_\Sigma$, h_Σ are assigned functions and $\partial_i\Omega$, $\partial_{i+1}\Omega$ $(i = 1, 3)$ denote complementary and disjoint subsets of $\partial\Omega$.

Each field above is assumed as smooth as requested to give sense to what is written.

Equations $(4)_{3,4}$, in which $\boldsymbol{\Gamma}$ and γ are assigned tensor and scalar fields, respectively, both of them positive definite, reveal the (energetically) *dissipative* character actually admitted for the boundary conditions of surface traction and equilibrated stress.

We shall call *kinematically admissible* for $\mathcal{B}$ any motion $(\boldsymbol{u}, \varphi)$ meeting the displacement boundary values $(4)_{1,2}$; when $(\boldsymbol{u}, \varphi)$ satisfies the whole of the equations (1), (2), (3), (4), it will be referred to as a (regular) *solution* to the problem at issue.

3. THE MINIMUM PRINCIPLE.

In view of constructing the functional to be minimized, let us introduce the following definitions (cf. [8]). From now on, we assume $T = +\infty$.

A (tensor, vector or scalar valued) field on Ω_T will be said *bounded at* ∞ if $\lim\limits_{t\to+\infty}$ of it exists $\forall \boldsymbol{x} \in \Omega$; of course, a (smooth) field bounded at ∞, say $\boldsymbol{u}(\boldsymbol{x}, t)$, admits Laplace-transform «$\hat{\ }$» in Ω:

$$\hat{\boldsymbol{u}}(\boldsymbol{x}, s) \equiv \int\limits_{(0,\infty)} \exp(-st)\, \boldsymbol{u}(\boldsymbol{x}, t)\, dt\,, \qquad s \in R^+ \equiv (0, +\infty)\,.$$

Moreover, we shall denote by $\mathcal{G}$ the set of the functions g: $t \in [0, +\infty) \to g(t) \in R^+$ such that:

$$\text{(i)}\qquad g(t) = \int\limits_{R^+} \exp(-st)\, G(s)\, ds$$

for some smooth positive function G defined on $s \in R^+$;

(ii) the integrals

$$\iint\limits_{R^+\times R^+} g(t+\tau)\, dt\, d\tau\,, \qquad \iint\limits_{R^+\times R^+} g'(t+\tau)\, dt\, d\tau\,, \qquad \iint\limits_{R^+\times R^+} g''(t+\tau)\, dt\, d\tau$$

are meaningful (g', g'' derivatives of g with respect to the bracketed argument).

Examples of $g \in \mathcal{G}$ can be found in [8].

Finally, let $\mathcal{H}$ be the class of all kinematically admissible motions $(\boldsymbol{v}, \psi)$ for $\mathcal{B}$ such that $\boldsymbol{v}$, ψ, $\nabla\boldsymbol{v}$, $\nabla\psi$, $\dot{\boldsymbol{v}}$, $\dot{\psi}$ are bounded at ∞.

We now prove the following

MINIMUM PRINCIPLE. *Let* $(\boldsymbol{u}, \varphi)$ *be a solution to the problem* (1), (2), (3), (4) *with all data beeing bounded at* ∞, *and let also* $\boldsymbol{u}$, φ, $\nabla\boldsymbol{u}$, $\nabla\varphi$, $\dot{\boldsymbol{u}}$, $\dot{\varphi}$ *be bounded at* ∞.

Given any $g \in \mathcal{G}$, *consider the following (well-defined) functional over* $\mathcal{H}$ (3):

$$\Phi[(\boldsymbol{v},\psi); g] = \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \left[\frac{1}{2}\rho\dot{\boldsymbol{v}}(t)\cdot\dot{\boldsymbol{v}}(\tau) + \frac{1}{2}\rho k\dot{\psi}(t)\,\dot{\psi}(\tau)\right]d\Omega\right\}dt\,d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \left[\frac{1}{2}\boldsymbol{C}\boldsymbol{F}(t):\boldsymbol{F}(\tau) + \frac{1}{2}\boldsymbol{A}\,\nabla\psi(t)\cdot\nabla\psi(\tau) + \frac{1}{2}\xi\psi(t)\,\psi(\tau) + \right.\right.$$

$$\left.\left. + \boldsymbol{D}\boldsymbol{F}(t)\cdot\nabla\psi(\tau) + \psi(t)\,\boldsymbol{B}:\boldsymbol{F}(\tau) + \psi(t)\boldsymbol{f}\cdot\nabla\psi(\tau)\right]d\Omega\right\}dt\,d\tau +$$

$$- \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega [\rho\boldsymbol{b}(t)\cdot\boldsymbol{v}(\tau) + \rho l(t)\,\psi(\tau)]\,d\Omega\right\}dt\,d\tau +$$

$$- \iint_{R^+\times R^+} g(t+\tau)\left\{\int_{\partial_2\Omega}\left[\boldsymbol{t}_\Sigma(t) - \frac{1}{2}\boldsymbol{\Gamma}\dot{\boldsymbol{v}}(t)\right]\cdot\boldsymbol{v}(\tau)\,d\Sigma + \int_{\partial_2\Omega}\left[h_\Sigma(t) - \frac{1}{2}\gamma\dot{\psi}(t)\right]\psi(\tau)\,d\Sigma\right\}dt\,d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \frac{1}{2}\omega\dot{\psi}(t)\,\psi(\tau)\,d\Omega\right\}dt\,d\tau +$$

$$+ \int_{R^+} g(t)\,dt\left\{\int_{\partial_2\Omega}\left[\frac{1}{2}\boldsymbol{v}(0) - \boldsymbol{u}_0\right]\cdot\boldsymbol{\Gamma}\boldsymbol{v}(t)\,d\Sigma + \int_{\partial_4\Omega}\left[\frac{1}{2}\psi(0) - \varphi_0\right]\gamma\psi(t)\,d\Sigma + \right.$$

$$\left. + \int_\Omega \omega\left[\frac{1}{2}\psi(0) - \varphi_0\right]\psi(t)\,d\Omega\right\} +$$

$$+ g(0)\int_\Omega \rho\left\{\boldsymbol{v}(0)\cdot\left[\frac{1}{2}\boldsymbol{v}(0) - \boldsymbol{u}_0\right] + k\psi(0)\left[\frac{1}{2}\psi(0) - \varphi_0\right]\right\}d\Omega +$$

$$+ \int_{R^+} g(t)\,dt\int_\Omega \rho\,\{[\boldsymbol{v}(0) - \boldsymbol{u}_0]\cdot\dot{\boldsymbol{v}}(t) - \dot{\boldsymbol{u}}_0\cdot\boldsymbol{v}(t) + k[\psi(0) - \varphi_0]\,\dot{\psi}(t) - k\dot{\varphi}_0\,\psi(t)\}\,d\Omega\,,$$

where $\boldsymbol{F} \equiv \frac{1}{2}[\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^t]$.

Then, it is

$$\Phi[(\boldsymbol{v},\psi); g] \geqslant \Phi[(\boldsymbol{u},\varphi); g] \qquad \forall(\boldsymbol{v},\psi) \in \mathcal{H},$$

and equality holds if and only if $(\boldsymbol{v},\psi) \equiv (\boldsymbol{u},\varphi)$.

PROOF. The proof consists in evaluating the difference

$$\Delta\Phi \equiv \Phi[(\boldsymbol{v},\psi); g] - \Phi[(\boldsymbol{u},\varphi); g]\,.$$

To this end, extensive recourse to reversal of order of integration between space and time and integration by parts for t and τ will be made (this is actually permitted by the properties of regularity of the relevant fields tacitly understood in the formulation of problem (1), (2), (3), (4)).

Note that $\boldsymbol{v} - \boldsymbol{u} \equiv \boldsymbol{0}$ on $\partial_1\Omega \times [0,T)$, $\psi - \varphi \equiv 0$ on $\partial_3\Omega \times [0,T)$.

(3) From now on, we explicitly indicate (only) the time-dependence of the fields in object.

After some lengthy calculations, it results:

$$\Delta\Phi = \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \left[\frac{1}{2}\rho(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(\tau)+\right.\right.$$

$$\left.\left.+\frac{1}{2}\rho k(\dot{\psi}-\dot{\varphi})(t)(\dot{\psi}-\dot{\varphi})(\tau)\right]d\Omega\right\}dt\,d\tau+$$

$$+\iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \left[\frac{1}{2}\boldsymbol{C}(\boldsymbol{F}-\boldsymbol{E})(t):(\boldsymbol{F}-\boldsymbol{E})(\tau)+\frac{1}{2}\boldsymbol{A}\,\nabla(\psi-\varphi)(t)\cdot\nabla(\psi-\varphi)(\tau)+\right.\right.$$

$$+\frac{1}{2}\xi(\psi-\varphi)(t)(\psi-\varphi)(\tau)+\boldsymbol{D}(\boldsymbol{F}-\boldsymbol{E})(t)\cdot\nabla(\psi-\varphi)(\tau)+(\psi-\varphi)\,\boldsymbol{B}:(\boldsymbol{F}-\boldsymbol{E})(\tau)+$$

$$\left.\left.+(\psi-\varphi)(t)\boldsymbol{f}\cdot\nabla(\psi-\varphi)(\tau)\right]d\Omega\right\}dt\,d\tau+$$

$$+\iint_{R^+\times R^+} g(t+\tau)\left\{\int_{\partial_2\Omega}\frac{1}{2}\Gamma(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)\cdot(\boldsymbol{v}-\boldsymbol{u})(\tau)\,d\Sigma+\int_{\partial_4\Omega}\frac{1}{2}\gamma(\dot{\psi}-\dot{\varphi})(t)(\psi-\varphi)(\tau)\,d\Sigma\right\}dt\,d\tau+$$

$$+\iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \frac{1}{2}\omega(\dot{\psi}-\dot{\varphi})(t)(\psi-\varphi)(\tau)\,d\Omega\right\}dt\,d\tau+$$

$$+\int_{R^+} g(t)\,dt\left\{\int_{\partial_2\Omega}\left(\left[\frac{1}{2}\boldsymbol{v}(0)-\boldsymbol{u}_0\right]\cdot\boldsymbol{\Gamma}\boldsymbol{v}(t)+\frac{1}{2}\boldsymbol{u}_0\cdot\Gamma\boldsymbol{u}(t)\right)d\Sigma+\right.$$

$$+\int_{\partial_4\Omega}\left(\left[\frac{1}{2}\psi(0)-\varphi_0\right]\gamma\psi(t)+\frac{1}{2}\varphi_0\,\gamma\varphi(t)\right)d\Sigma+$$

$$\left.+\int_\Omega\left(\omega\left[\frac{1}{2}\psi(0)-\varphi_0\right]\psi(t)+\frac{1}{2}\omega\varphi_0\,\varphi(t)\right)d\Omega\right\}+$$

$$+g(0)\int_\Omega \rho\left\{\frac{1}{2}[\boldsymbol{v}(0)-\boldsymbol{u}_0]^2+\frac{1}{2}k[\psi(0)-\varphi_0]^2\right\}d\Omega+$$

$$+\iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \rho[\dot{\boldsymbol{u}}(t)\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(\tau)+k\dot{\varphi}(t)(\dot{\psi}-\dot{\varphi})(\tau)]\,d\Omega)\right\}dt\,d\tau+$$

$$+\iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega [\boldsymbol{C}\boldsymbol{E}(t):(\boldsymbol{F}-\boldsymbol{E})(\tau)+\boldsymbol{A}\,\nabla\varphi(t)\cdot\nabla(\psi-\varphi)(\tau)+\xi\varphi(t)(\psi-\varphi)(\tau)+\right.$$

$$+\boldsymbol{D}\boldsymbol{E}(t)\cdot\nabla(\psi-\varphi)(\tau)+\boldsymbol{D}\,\nabla\varphi(t):(\boldsymbol{F}-\boldsymbol{E})(\tau)+\varphi(t)\boldsymbol{B}:(\boldsymbol{F}-\boldsymbol{E})(\tau)+\boldsymbol{B}:\boldsymbol{E}(t)(\psi-\varphi)(\tau)+$$

$$\left.+\varphi(t)\boldsymbol{f}\cdot\nabla(\psi-\varphi)(\tau)+\boldsymbol{f}\cdot\nabla\varphi(t)(\psi-\varphi)(\tau)\,d\Omega\right\}dt\,d\tau+$$

$$-\iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \rho[\boldsymbol{b}(t)\cdot(\boldsymbol{v}-\boldsymbol{u})(\tau)+l(t)(\psi-\varphi)(\tau)]\,d\Omega\right\}dt\,d\tau+$$

$$- \iint_{R^+\times R^+} g(t+\tau)\left\{ \int_{\partial_2\Omega}\left(\left[\boldsymbol{t}_\Sigma(t)-\frac{1}{2}\Gamma\dot{\boldsymbol{u}}(t)\right]\cdot(\boldsymbol{v}-\boldsymbol{u})(\tau)-\frac{1}{2}\Gamma\boldsymbol{u}(t)\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(\tau)\right)d\Sigma + \right.$$

$$\left. + \int_{\partial_4\Omega}\left(\left[h_\Sigma(t)-\frac{1}{2}\gamma\dot{\varphi}(t)\right](\psi-\varphi)(\tau)-\frac{1}{2}\gamma\varphi(t)(\dot{\psi}-\dot{\varphi})(\tau)\right)d\Sigma\right\} dt\, d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega \frac{1}{2}\omega[\dot{\varphi}(t)(\psi-\varphi)(\tau)+\varphi(t)(\dot{\psi}-\dot{\varphi})(\tau)]\,d\Omega\right\}dt\,d\tau +$$

$$+ \int_{R^+} g(t)\,dt \int_\Omega \rho\{[\boldsymbol{v}(0)-\boldsymbol{u}_0]\cdot\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{u}}_0\cdot(\boldsymbol{v}-\boldsymbol{u})(t)+$$

$$+ k[\psi(0)-\varphi_0]\,\dot{\psi}(t)-k\dot{\varphi}_0(\psi-\varphi)t\}\,d\Omega\,.$$

Consider now that $(\boldsymbol{u},\varphi)$ is a solution, and multiply $(1)_1$ and $(1)_2$, evaluated at $t\in(0,+\infty)$, for $(\boldsymbol{v}-\boldsymbol{u})$ and $(\psi-\varphi)$, respectively, evaluated at $\tau\in(0,+\infty)$. The sum of the last six integrals comes out to be simply equal to:

$$\int_{R^+} g(t)\,dt\left\{\int_{\partial_2\Omega}\frac{1}{2}(\Gamma(\boldsymbol{v}-\boldsymbol{u})(t)\cdot\boldsymbol{u}_0-\Gamma\boldsymbol{u}(t)\cdot[\boldsymbol{v}(0)-\boldsymbol{u}_0])\,d\Sigma + \right.$$

$$+ \int_{\partial_4\Omega}\frac{1}{2}(\gamma(\psi-\varphi)(t)\,\varphi_0-\gamma\varphi(t)[\psi(0)-\varphi_0])\,d\Sigma +$$

$$\left. + \int_\Omega \frac{1}{2}(\omega(\psi-\varphi)(t)\,\varphi_0-\omega\varphi(t)[\psi(0)-\varphi_0])\,d\Omega\right\} +$$

$$+ \int_{R^+} g(t)\,dt\int_\Omega \rho\{[\boldsymbol{v}(0)-\boldsymbol{u}_0]\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)+k[\psi(0)-\varphi_0](\dot{\psi}-\dot{\varphi})(t)\}\,d\Omega\,.$$

We thus have:

$$\Delta\Phi = \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega\left[\frac{1}{2}\rho(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(\tau)+\frac{1}{2}\rho k(\dot{\psi}-\dot{\varphi})(t)(\dot{\psi}-\dot{\varphi}(\tau)\right]d\Omega\right\}dt\,d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega\left[\frac{1}{2}\boldsymbol{C}(\boldsymbol{F}-\boldsymbol{E})(t):(\boldsymbol{F}-\boldsymbol{E})(\tau)+\frac{1}{2}\boldsymbol{A}\,\nabla(\psi-\varphi)(t)\cdot\nabla(\psi-\varphi)(\tau)+\right.\right.$$

$$+\frac{1}{2}\xi(\psi-\varphi)(t)(\psi-\varphi)(\tau)+\boldsymbol{D}(\boldsymbol{F}-\boldsymbol{E})(t)\cdot\nabla(\psi-\varphi)(\tau)+$$

$$\left.\left.+(\psi-\varphi)\boldsymbol{B}:(\boldsymbol{F}-\boldsymbol{E})(\tau)+(\psi-\varphi)(t)\boldsymbol{f}\cdot\nabla(\psi-\varphi)(\tau)\right]d\Omega\right\}dt\,d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_{\partial_2\Omega}\frac{1}{2}\Gamma(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)\cdot(\boldsymbol{v}-\boldsymbol{u})(\tau)\,d\Sigma+\int_{\partial_4\Omega}\frac{1}{2}\gamma(\dot{\psi}-\dot{\varphi})(t)(\psi-\varphi)(\tau)\,d\Sigma\right\}dt\,d\tau +$$

$$+ \iint_{R^+\times R^+} g(t+\tau)\left\{\int_\Omega\frac{1}{2}\omega(\dot{\psi}-\dot{\varphi})(t)(\psi-\varphi)(\tau)\,d\Omega\right\}dt\,d\tau +$$

$$+ \int_{R^+} g(t)\,dt\left\{\int_{\partial_2\Omega}\frac{1}{2}\Gamma(\boldsymbol{v}-\boldsymbol{u})(t)\cdot[\boldsymbol{v}(0)-\boldsymbol{u}_0]\,d\Sigma+\right.$$

$$+\int_{\partial_4\Omega}\frac{1}{2}\gamma(\psi-\varphi)(t)[\psi(0)-\varphi_0]\,d\Sigma+\int_{\Omega}\frac{1}{2}\omega(\psi-\varphi)(t)[\psi(0)-\varphi_0]\,d\Omega\Bigg\}+$$

$$+\int_{R^+}g(t)\,dt\int_{\Omega}\rho\{[\boldsymbol{v}(0)-\boldsymbol{u}_0]\cdot(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})(t)+k[\psi(0)-\varphi_0](\dot{\psi}-\dot{\varphi})(t)\}\,d\Omega+$$

$$+g(0)\int_{\Omega}\rho\left\{\frac{1}{2}[\boldsymbol{v}(0)-\boldsymbol{u}_0]^2+\frac{1}{2}k[\psi(0)-\varphi_0]^2\right\}d\Omega\,.$$

Insert now the expression for g (see (i) on p. 190) in the foregoing equation; Laplace-transform «ˆ» of the fields will appear, so that we finally obtain:

$$\Delta\Phi=\int_{R^+}G(s)\,ds\Bigg\{\int_{\Omega}\Bigg[\frac{1}{2}\rho s^2(\hat{\boldsymbol{v}}-\hat{\boldsymbol{u}})^2+\frac{1}{2}\rho k s^2(\hat{\psi}-\hat{\varphi})^2+$$

$$+\frac{1}{2}\boldsymbol{C}(\hat{\boldsymbol{F}}-\hat{\boldsymbol{E}}):(\hat{\boldsymbol{F}}-\hat{\boldsymbol{E}})+\frac{1}{2}\boldsymbol{A}\,\nabla(\hat{\psi}-\hat{\varphi})\cdot\nabla(\hat{\psi}-\hat{\varphi})+\frac{1}{2}\xi(\hat{\psi}-\hat{\varphi})^2+$$

$$+\boldsymbol{D}(\hat{\boldsymbol{F}}-\hat{\boldsymbol{E}})\cdot\nabla(\hat{\psi}-\hat{\varphi})+(\hat{\psi}-\hat{\varphi})\boldsymbol{B}:(\hat{\boldsymbol{F}}-\hat{\boldsymbol{E}})+(\hat{\psi}-\hat{\varphi})\boldsymbol{f}\cdot\nabla(\hat{\psi}-\hat{\varphi})+\frac{1}{2}\omega s(\hat{\psi}-\hat{\varphi})^2\Bigg]d\Omega+$$

$$+\int_{\partial_2\Omega}\frac{1}{2}s\Gamma(\hat{\boldsymbol{v}}-\hat{\boldsymbol{u}})\cdot(\hat{\boldsymbol{v}}-\hat{\boldsymbol{u}})\,d\Sigma+\int_{\partial_4\Omega}\frac{1}{2}s\gamma(\hat{\psi}-\hat{\varphi})^2\,d\Sigma\Bigg\}\,.$$

On recalling the assumptions of positive definiteness previously made, the thesis is then achieved.

REMARK. The principle just proven contains the dynamical counterpart of the *minimum potential energy principle* given by lesan in [7] for the equilibrium problem. Of course, as a corollary, one recovers from it the uniqueness of regular solutions (cf. [2]).

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