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On lattice automorphisms of the special linear group

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Teoria dei gruppi. — *On lattice automorphisms of the special linear group* (*). Nota (**) di MAURO COSTANTINI (***), presentata dal Corrisp. G. ZACHER.

ABSTRACT. — We show, with a counterexample, that proposition 3 in [2], as it stands, is not correct; we prove however that by changing the hypothesis the thesis of the proposition remains still valid.

KEY WORDS: Linear groups; Lattice automorphisms.

RIASSUNTO. — *Sugli automorfismi reticolari del gruppo lineare speciale.* Nella presente Nota si stabilisce, mediante un controesempio, che la Prop. 3 in [2] ed il relativo corollario sono errati; si prova che, modificando opportunamente le ipotesi colà espresse, la tesi sostenuta risulta corretta.

During the work on our Ph. D. thesis, we came across a problem similar to one dealt with by H. Völklein in [2]. In the present paper we point out that Prop. 3 with its corollary in [2] is not correct, giving a counterexample. However we are able, by modifying the assumptions, to show that the thesis stated in the above mentioned proposition, is still valid.

§ 1. In this section we construct the above mentioned counterexample. Let's consider the group $G = SL(3,27)$, and denote by Aut $\mathcal{L}(G)$ the group of autoprojectivities of G, and by Φ the subgroup of the autoprojectivities which fix every 3-Sylow subgroup of G. Identifying Aut G with the subgroup of autoprojectivities of G induced by automorphisms, we have, by Prop. 2 in [2], Aut $\mathcal{L}(G) = \Phi \rtimes$ Aut G. We'll show that G is strongly lattice determined, that is that Φ is the identity subgroup.

We denote by T the subgroup of diagonal matrices of G. From the corollary on page 11 of [3], to prove that $\Phi = \{1\}$ it is enough to show that $X^\varphi = X$ for every subgroup X of T and every φ in Φ. In our case T is isomorphic to $C_{26} \times C_{26}$, and so, by Lemma 1 in [3], it is enough to show that every φ in Φ fixes every subgroup of order 13 of T. Let $\mathcal{M}$ be the set of the 14 subgroups of order 13 of T. If we make the Weyl group $W = N(T)/T$ act naturally on $\mathcal{M}$, we get four orbits, of which two have three elements, one has two elements and one has six elements. Let's call this last orbit $\mathcal{E}$. We now fix an element φ of Φ. As we showed in [1], φ fixes every subgroup of T which lies in the orbits with two or three elements. To prove that the same holds for the elements of $\mathcal{E}$, we need a description of these subgroups. Let u be a fixed element of order 13 in $\boldsymbol{F}_{27}^{\times}$, let e be the element diag (u, u^{10}, u^2) of T, and $E = \langle e \rangle$. Then we have that E lies in $\mathcal{E}$ and for every E' in $\mathcal{E}$ there exists a unique w in W such that $E' = E^w$. Now let P be the following subgroup of the unitriangular upper matrices:

$$P = \left\{ \begin{pmatrix} 1 & k & k^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| k \in \boldsymbol{F}_{27} \right\}.$$

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From a direct calculation, it follows that E is contained in $N(P)$, and even that E is the unique subgroup in $\mathcal{E}$ satisfying this condition. Also, we observe that E^φ lies in $\mathcal{E}$, as φ is index preserving and we already know that it fixes all the subgroups of order 13 of T out of $\mathcal{E}$. Now P is the unique 3-Sylow subgroup of the group EP, thus P^φ is the unique 3-Sylow subgroup of $\langle E^\varphi, P^\varphi \rangle$. Hence we get that E^φ is contained in $N(P^\varphi)$. Finally, as by Prop 1 in [3] P is fixed by φ, we have $E^\varphi \leq N(P)$, which gives $E^\varphi = E$, since E is the unique subgroup in $\mathcal{E}$ contained in $N(P)$. If now E' is any element of $\mathcal{E}$, we just need to take any g in $N(T)$ such that $E' = E^g$, and apply the previous arguments to the groups E' and P^g. Thus Φ is the identity group, and G is strongly lattice determined.

Now we'll construct for every w in W an autoprojectivity φ_w of T satisfying the conditions of Prop. 3 in [2]. So let w be a fixed element of W. We define φ_w to be the identity on the 2-subgroups of T and on the subgroups of order 13 which lie in orbits of length two or three. If E' is any element in $\mathcal{E}$, there exists a unique ρ in W such $E' = E^\rho$. We then put $E'^{\varphi_w} = E^{w\rho}$. There exists a unique way to extend φ_w to an autoprojectivity of T. From a direct calculation it is possible to show that, for every w in W, φ_w satisfies the conditions i)-iii) of Prop. 3 (and also that these are all). From the fact that 13 does not divide $3^r - 1$ for $r = 1$ or 2, it follows that φ_w satisfies also condition iv) for every w in W. Finally we can say that for every non trivial w in W we have a non trivial autoprojectivity φ_w of T which satisfies the hypothesis of Prop. 3, but which does not fit with the thesis, because we already know that $\Phi = \{1\}$. Besides, by taking the prime l equal 13, we see that G represents a counterexample also for the corollary following Prop. 3 in [2].

The point is that if λ satisfies the hypothesis of Prop. 3, X is a subgroup of T not fixed by λ and P is a p-subgroup of $SL(3, p^\nu)$ such that $X \leq N(P)$, then condition iv) is not enough to guarantee that $X^\lambda \leq N(P)$. In the following paragraph we are going to modify the content of condition iv).

§2. First we give some notation. Let p be any prime, q a power of p: we put $K = \boldsymbol{F}_q$ and $G = SL(3, K)$. We denote by T the subgroup of diagonal matrices of G and by U the subgroup of upper unitriangular matrices of G. Also, if F is a subfield of K we denote by $T(F)$ the subgroup of T whose elements have entries in F.

Let $s = \operatorname{diag}(\alpha, \beta, \gamma)$ be an element of T. Then we put:

$$x_1(s) = \alpha\beta^{-1}, \quad x_2(s) = \beta\gamma^{-1}, \quad x_3(s) = \alpha\gamma^{-1} \quad (= x_1(s)\, x_2(s)) .$$

We'll just write x_i for $x_i(s)$ when there is no ambiguity. Also, for $i = 1, 2, 3$, we denote by $\mu_i(s)$ the minimum polynomial of $x_i(s)$ over $\boldsymbol{F}_p$. Again we will just write μ_i for $\mu_i(s)$ when there is no ambiguity.

Definition. We say that an element s in T satisfies $(*)$ if:

i) $|s| = |x_i(s)|$ for every $i = 1, 2, 3$; (which implies that $\deg \mu_i = \deg \mu_j$ for every $i, j = 1, 2, 3$);

ii) $\mu_i \neq \mu_j$ for every $i \neq j$.

Definition. We say that an element s in T satisfies $(**)$ if nsn^{-1} satisfies $(*)$ for every n in $N(T)$.

Suppose that s in T satisfies $(*)$. Then we have $\boldsymbol{F}_p(x_1) = \boldsymbol{F}_p(x_2) = \boldsymbol{F}_p(x_3) = \boldsymbol{F}_{p^n}$, where

n is the degree of the μ_i's. We'll denote this subfield of K by $F(s)$, and simply by F if there is no ambiguity.

PROP. 2.1. Let s be an element of T satisfying $(*)$. Then the map

$$\Psi: \boldsymbol{F}_p[X] \to F \times F \times F \quad \text{given by} \quad \theta \to (\theta(x_1), \theta(x_2), \theta(x_3))$$

is a surjective $\boldsymbol{F}_p$-algebra homomorphism.

PROOF. It's clear that Ψ is an $\boldsymbol{F}_p$-algebra homomorphism. To show that Ψ is surjective we observe that condition $(*)$ implies that ket $\Psi = (\mu_1 \mu_2 \mu_3)$. Hence we have an induced $\boldsymbol{F}_p$-algebra monomorphism

$$\overline{\Psi}: \boldsymbol{F}_p[X]/(\mu_1 \mu_2 \mu_3) \to F \times F \times F .$$

But now $\boldsymbol{F}_p[X]/(\mu_1 \mu_2 \mu_3)$ and $F \times F \times F$ are both $\boldsymbol{F}_p$-spaces of dimension $3n$, where n is $\deg \mu_i$. Hence $\overline{\Psi}$ is an isomorphism, and Ψ is surjective. #

We now consider the three monomorphisms $\chi_i: K \to U$ defined by:

$$\chi_1(k) = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \chi_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \qquad \chi_3(k) = \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

We therefore have the commutator formula:

$$\chi_2(b)\chi_1(a) = \chi_1(a)\chi_2(b)\chi_3(ab) \qquad \text{for every } a,\ b \text{ in } K.$$

Also, from the fact that $\operatorname{Im}\chi_i$, for $i = 1, 2, 3$, are exactly the three root-subgroups of T contained in U, for every u in U there exists a unique 3-tuple (a, b, c) with a, b, c in K such that $u = \chi_1(a)\chi_2(b)\chi_3(c)$.

In the following we fix an element u in U and hence three elements a, b, c of K such that $u = \chi_1(a)\chi_2(b)\chi_3(c)$.

PROP. 2.2. For every n in $\mathbb{Z}$ there exists $h(n)$ in $\mathbb{Z}$ such that

$$u^n = \chi_1(na)\chi_2(nb)\chi_3(nc + h(n)ab) .$$

PROOF. We can take $h(n) = 0$ if $n = 0, 1$; $h(n) = n(n+1)/2$ if $n \geq 2$ and $h(n) = (-n+1)(-n+2)/2$ if $n < 0$. The result then comes by induction. #

We now fix s in T such that s satisfies $(*)$.

PROP. 2.3. Let θ be an element of $\boldsymbol{F}_p[X]$. Then there exists γ in F such that

$$\chi_1(\theta(x_1)a)\chi_2(\theta(x_2)b)\chi_3(\theta(x_3)c + \gamma ab) \qquad \text{is in } \langle u \rangle^{\langle s \rangle} .$$

PROOF. We use induction on $\deg \theta$. Suppose $\deg \theta = 0$, then $\theta = k$ is in $\boldsymbol{F}_p$. Choose n in $\mathbb{Z}$ such that $n \to k$ under the natural map $\pi: \mathbb{Z} \to K$ given by $m \to m \cdot 1_K$ for every m in $\mathbb{Z}$. Then, from Prop. 2.2, we get $h(n)$ in $\mathbb{Z}$ such that

$$u^n = \chi_1(na)\chi_2(nb)\chi_3(nc + h(n)ab) .$$

But then

$$\chi_1(ka)\chi_2(kb)\chi_3(kc + (h(n) \cdot 1_K)ab) = u^n \qquad \text{is in } \langle u \rangle^{\langle s \rangle} ,$$

$k = \theta(x_i)$ for every $i = 1,2,3$ and $h(n)\cdot 1_K$ is in F. So now assume the result for all θ in $\boldsymbol{F}_p[X]$ with $\deg\theta \leq r$ and let θ be of degree $r+1$. Then $\theta = \theta' + k_{r+1}X^{r+1}$, where θ' has degree $\leq r$, and k_{r+1} is in $\boldsymbol{F}_p$. Then, by induction, there exists γ' in F such that

$$v = \chi_1(\theta'(x_1)\,a)\,\chi_2(\theta'(x_2)\,b)\,\chi_3(\theta'(x_3)\,c + \gamma' ab) \qquad \text{is in } \langle u\rangle^{\langle s\rangle}.$$

Choose n_{r+1} in $\mathbb{Z}$ such that $\pi(n_{r+1}) = k_{r+1}$, and let $w = (s^{r+1}us^{-(r+1)})^{n_{r+1}}$. Then we have $w \in \langle u\rangle^{\langle s\rangle}$. Also we have

$$w = \chi_1(n_{r+1}x_1^{r+1}a)\,\chi_2(n_{r+1}x_2^{r+1}b)\,\chi_3(n_{r+1}x_3^{r+1}c + h(n_{r+1})\,x_3^{r+1}ab) \qquad \text{as } x_3 = x_1x_2.$$

Let $\gamma'' = h(n_{r+1})\,x_3^{r+1}$, so that γ'' is in F. Consider now the element vw, which is in $\langle u\rangle^{\langle s\rangle}$. Using the commutator formula, we have

$$vw = \chi_1((\theta'(x_1) + n_{r+1}x_1^{r+1})\,a)\,\chi_2((\theta'(x_2) + n_{r+1}x_2^{r+1})\,b)\cdot$$
$$\cdot\chi_3(\theta'(x_3) + n_{r+1}x_3^{r+1})\,c + (\gamma' + \gamma'' + n_{r+1}x_1^{r+1}\theta'(x_2))\,ab).$$

which gives the result we want, by defining

$$\gamma = \gamma' + \gamma'' + n_{r+1}x_1^{r+1}\theta'(x_2)$$

and noting that $\theta'(x_i) + n_{r+1}x_i^{r+1} = \theta(x_i)$ for every $i = 1,2,3$. #

Prop. 2.4. For every A, B, C in F there exists k in F such that

$$\chi_1(Aa)\,\chi_2(Bb)\,\chi_3(Cc + kab) \qquad \text{is in} \langle u\rangle^{\langle s\rangle}.$$

Proof. By Prop. 2.1, there exists θ in $\boldsymbol{F}_p[X]$ such that $\theta(x_1) = A$, $\theta(x_2) = B$ and $\theta(x_3) = C$. Then, by Prop. 2.3, there exists γ in F such that

$$\chi_1(\theta(x_1)\,a)\,\chi_2(\theta(x_2)\,b)\,\chi_3(\theta(x_3)\,c + \gamma ab) \qquad \text{is in} \langle u\rangle^{\langle s\rangle}.$$

So we just need to take $k = \gamma$ to get the result. #

Prop. 2.5. Let D be in F. Then $\chi_3(Dab)$ lies in $\langle u\rangle^{\langle s\rangle}$.

Proof. Suppose we have $\xi_1, \xi_2, \zeta, \zeta'$ in K and let

$$y = \chi_1(\xi_1)\,\chi_2(\xi_2)\,\chi_3(\zeta), \qquad y' = \chi_1(\xi_1)\,\chi_2(\xi_2)\,\chi_3(\zeta').$$

Then we have

$$yy'^{-1} = \chi_1(\xi_1)\,\chi_2(\xi_2)\,\chi_3(\zeta)\,\chi_3(-\zeta')\,\chi_2(-\xi_2)\,\chi_1(-\xi_1) = \chi_3(\zeta - \zeta')$$

as $\operatorname{Im}\chi_3 = Z(U)$.

We apply this to the following:

let A, A', B, B' be elements of F. From Prop. 2.4 there exist k, k' in F such that

$$v = \chi_1(Aa)\,\chi_2(Bb)\,\chi_3(kab) \qquad \text{and} \qquad w = \chi_1(A'a)\,\chi_2(B'b)\,\chi_3(k'ab)$$

are both in $\langle u\rangle^{\langle s\rangle}$. Then vw and wv are both in $\langle u\rangle^{\langle s\rangle}$, and we have:

$$vw = \chi_1((A + A')\,a)\,\chi_2((B + B')\,b)\,\chi_3((k + k' + A'B)\,ab),$$
$$wv = \chi_1((A + A')\,a)\,\chi_2((B + B')\,b)\,\chi_3((k + k' + AB')\,ab).$$

Hence $\chi_3((A'B - AB')\,ab) = (vw)(wv)^{-1}$ lies in $\langle u\rangle^{\langle s\rangle}$.

Finally, if we let $A' = D$, $B = 1$, $A = B' = 0$, then we obtain that $\chi_3(Dab)$ lies in $\langle u\rangle^{\langle s\rangle}$, as we wanted. #

PROP. 2.6. Let A, B, C, D be elements of F. Then

$$\chi_1(Aa)\,\chi_2(Bb)\,\chi_3(Cc + Dab) \qquad \text{is in } \langle u\rangle^{\langle s\rangle}.$$

PROOF. From Prop. 2.4, there exists k in F such that

$$v = \chi_1(Aa)\,\chi_2(Bb)\,\chi_3(Cc + kab) \qquad \text{is in } \langle u\rangle^{\langle s\rangle}.$$

Then take $k' = D - k$ and apply Prop. 2.5 to obtain that $w = \chi_3(k'ab)$ is in $\langle u\rangle^{\langle s\rangle}$. Hence $\chi_1(Aa)\,\chi_2(Bb)\,\chi_3(Cc)\,\chi_3(Cc + Dab) = vw$ lies in $\langle u\rangle^{\langle s\rangle}$. #

We are now able to state the result that we'll use in the final step.

LEMMA. Let s be an element of T satisfying $(*)$. Then, if P is a subgroup of U such that s is in $N(P)$, we have that $T(F(s))$ is contained in $N(P)$.

PROOF. Let t be in $T(F(s))$, i.e. $t = \mathrm{diag}(\alpha, \beta, \gamma)$, with α, β, γ in $F(s)$. Then, if $u = \chi_1(a)\,\chi_2(b)\,\chi_3(c)$ is in P, we have $tut^{-1} = \chi_1(\alpha\beta^{-1}a)\,\chi_2(\beta\gamma^{-1}b)\,\chi_3(\alpha\gamma^{-1}c)$ which is in $\langle u\rangle^{\langle s\rangle}$ by Prop. 2.6. Hence, for every u in P and for every t in $T(F(s))$, we have $tut^{-1} \in \langle u\rangle^{\langle s\rangle} \leq P^{\langle s\rangle} = P$. So for every t in $T(F(s))$ we have $P^t \leq P$, which implies that $T(F(s))$ is contained in $N(P)$. #

We are now in the position to prove the following statement, which represents the announced modification of Prop. 3 in [2]:

Let G be the group $SL(3, p^\nu)$, and let λ be a lattice automorphism of the group T of diagonal matrices in G. Then λ can be extended to a lattice automorphism of G fixing every p-subgroup of G and commuting with the inner automorphisms of G, if the following holds:

i) λ commutes with the action of $N(T)/T$;

ii) λ fixes every subgroup of T which is fixed by a non-trivial element of $N(T)/T$;

iii) λ fixes every 2-subgroup and every 3-subgroup of T;

vi′) if s is an element of prime power order of T not satisfying $(**)$, then λ fixes the group generated by s.

PROOF. We'll prove that if P is a p-subgroup of G normalized by a subgroup X of T, then also X^λ and $X^{\lambda^{-1}}$ normalize P. It is then possible to follow the proof of Prop. 3 in [2] from step (5) to get the result. So let X be a subgroup of T and P be a p-subgroup of G normalized by X. If X is fixed by λ, then there is nothing to prove. So assume that λ doesn't fix X. Without loss of generality we may assume that X is cyclic of prime power order. By condition iv′), there exists a generator s of X satisfying $(**)$. We can apply the same argument of step (4) in the proof of Prop. 3, to obtain an element n in $N(T)$ such that nPn^{-1} is contained in U. Now nsn^{-1} satisfies $(*)$, and so we can apply the lemma to get $T(F(nsn^{-1})) \leq N(nPn^{-1})$. Then we have that $nX^\lambda n^{-1}$ and $nX^{\lambda^{-1}}n^{-1}$ are both contained in $nN(P)\,n^{-1}$, as $T(F(nsn^{-1}))$ contains every subgroup of order $|s|$ of T, and λ is index-preserving. Hence we have that both X^λ and $X^{\lambda^{-1}}$ are contained in $N(P)$, and we are done. #

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