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A theorem on the controllability of pertubated linear control systems

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Analisi matematica. — *A theorem on the controllability of pertubated linear control systems.* Nota di Ornella Naselli Ricceri, presentata (*) dal socio G. Scorza Dragoni.

Abstract. — In this Note, applying our recent Theorem 3.1 of [7], we prove that suitable perturbations of a completely controllable linear control system, do not affect the controllability of the system.

Key words. — Linear control system; Controllability; Perturbation.

Riassunto. — *Un teorema sulla controllabilità di sistemi lineari di controllo perturbati.* In questa Nota, applicando il recente Teorema 3.1 di [7], provo che opportune perturbazioni di un sistema lineare di controllo completamente controllabile, non inficiano la controllabilità del sistema.

The purpose of this Note is to present an application of Theorem 3.1 of [7] to control theory, as announced in the same [7].

Our notation is standard. In any case, we refer to [1], [3]. In the sequel, n, m are two positive integers and $\mathbb{R}^{n,m}$ denotes the space of all real $n \times m$-matrices. We will consider the spaces $\mathbb{R}^n$, $\mathbb{R}^m$ endowed with their Euclidean norms. The norm of a matrix $C \in \mathbb{R}^{n,m}$ is denoted by $\|C\|$, that is $\|C\| = \sup\{\|Cu\|_{\mathbb{R}^n}\colon\ u \in \mathbb{R}^m,\ \|u\|_{\mathbb{R}^m} \leq 1\}$. The symbol $\mathfrak{M}([a,b], \mathbb{R}^{n,m})$ denotes the space of all measurable functions $B\colon [a,b] \to \mathbb{R}^{n,m}$.

On a given compact interval $[a,b] \subseteq \mathbb{R}$, consider the linear control system

$$\dot{x} = A(t)\,x + B(t)\,u \tag{1}$$

where $\dot{x} = dx/dt$, A is a function from $[a,b]$ into $\mathbb{R}^{n,n}$ and B is a function from $[a,b]$ into $\mathbb{R}^{n,m}$.

We say that the system (1) is completely controllable if for every $x_0, x_1 \in \mathbb{R}^n$, there exist $u \in L^\infty([a,b], \mathbb{R}^m)$ and $\varphi \in AC([a,b], \mathbb{R}^n)$ such that

$$\begin{cases} \dot{\varphi}(t) = A(t)\,\varphi(t) + B(t)\,u(t) & \text{a.e. in } [a,b], \\ \varphi(a) = x_0, \\ \varphi(b) = x_1. \end{cases}$$

Suppose that the system (1) is completely controllable. Then a natural stability question arises: if $\widetilde{A}$, $\widetilde{B}$ are two other matrix-valued functions on $[a,b]$, which are «close» to A, B, in a sense to be specified, is the new system

$$\dot{x} = \widetilde{A}(t)\,x + \widetilde{B}(t)\,u \tag{2}$$

completely controllable?

An answer to this question is provided, for instance, by the following well-known result of J. P. Dauer ([2], Theorem 4; see also Theorem 7.10.1 of [1]):

Theorem A. *Let $A \in L^1([a,b], \mathbb{R}^{n,n})$, $B \in L^1([a,b], \mathbb{R}^{n,m})$ and let the system* (1) *be completely controllable. Then, there exists $\rho > 0$ such that, for every $\widetilde{A} \in L^1([a,b], \mathbb{R}^{n,n})$,*

(*) Nella seduta del 13 maggio 1989.

$\widetilde{B} \in L^1([a,b], \mathbb{R}^{n,m})$ *satisfying*

$$\int_a^b \|\widetilde{A}(t) - A(t)\|\, dt + \int_a^b \|\widetilde{B}(t) - B(t)\|\, dt < \rho\,,$$

the system (2) *is completely controllable.*

The result we want to prove here is the following improvement of Theorem A:

Theorem 1. *Let* $A \in L^1([a, b], \mathbb{R}^{n,n})$, $B \in \mathcal{M}([a, b], \mathbb{R}^{n,m})$ *and let the system* (1) *be completely controllable. Then, there exists* $\rho > 0$ *such that, for every* $\widetilde{A} \in L^1([a,b], \mathbb{R}^{n,n})$, $\widetilde{B} \in \mathcal{M}([a,b], \mathbb{R}^{n,m})$ *satisfying*

$$\int_a^b \|\widetilde{A}(t) - A(t)\|\, dt + \int_a^b \frac{\|\widetilde{B}(t) - B(t)\|}{1 + \|\widetilde{B}(t) - B(t)\|}\, dt < \rho\,, \tag{3}$$

the system (2) *is completely controllable.*

Proof. We refer to [7] for what concerning multifunctions we will use. Assume that the conclusion of the theorem does not hold. Then, there are a sequence $\{A_k\}$ in $L^1([a,b], \mathbb{R}^{n,n})$ and a sequence $\{B_k\}$ in $\mathcal{M}([a,b], \mathbb{R}^{n,m})$ such that, for each $k \in \mathbb{N}$, one has

$$\int_a^b \|A_k(t) - A(t)\|\, dt + \int_a^b \frac{\|B_k(t) - B(t)\|}{1 + \|B_k(t) - B(t)\|}\, dt < \frac{1}{k} \tag{4}$$

and the system

$$\dot{x} = A_k(t)\, x + B_k(t)\, u$$

is not completely controllable.

From (4) it follows that the sequence $\{\|A_k(\cdot) - A(\cdot)\|\}$ converges to zero in $L^1[a, b]$) and that the sequence $\{\|B_k(\cdot) - B(\cdot)\|\}$ converges in measure to zero (see, for instance, [5], pp. 5-6). Hence, taking into account, for instance, Theorem 2.8.1 of [6], there are an increasing sequence $\{k_r\}$ in $\mathbb{N}$ and a function $g \in L^1([a, b])$ such that

$$\lim_{r \to \infty} \|A_{k_r}(t) - A(t)\| + \|B_{k_r}(t) - B(t)\| = 0 \quad \text{and } \|A_{k_r}(t)\| \le g(t) \tag{5}$$

a.e. in $[a, b]$, for all $r \in \mathbb{N}$.

Let Λ be the one-point compactification of $\{k_r\}$, with the usual topology. For every $t \in [a,b]$, $x \in \mathbb{R}^n$, $\lambda \in \Lambda$, put:

$$F(t,x,\lambda) = \begin{cases} A_{k_r}(t)\, x + B_{k_r}(t)\, \Gamma & \text{if } \lambda < \infty\,,\ \lambda = k_r \\ A(t)\, x + B(t)\, \Gamma & \text{if } \lambda = \infty\,, \end{cases}$$

where Γ is the closed unit ball of $\mathbb{R}^m$.

Of course, each set $F(t,x,\lambda)$ is closed. Observe that, for every $x \in \mathbb{R}^n$, $\lambda \in \Lambda$, the multifunction $F(\cdot\,, x, \lambda)$ is measurable (see, for instance, Theorem 6.5 of [4]). Moreover, taking into account (5), it is seen at once that, for almost every $t \in [a, b]$ and every $\lambda \in \Lambda$, the multifunction $F(t, \cdot, \lambda)$ is Lipschitzian, with Lipschitz constant $g(t)$. Further, for almost every $t \in [a,b]$ and every $x \in \mathbb{R}^n$, the multifunction $F(t, x, \cdot)$ is lower semicontinuous, thanks to (5). Therefore, taking also into account that $\theta_{\mathbb{R}^n} \in F(t, \theta_{\mathbb{R}^n}, \lambda)$ ($\theta_{\mathbb{R}^n}$ is the origin of $\mathbb{R}^n$), we are allowed to apply Theorem 3.1 of [7]. For each $\lambda \in \Lambda$, put

$$\mathcal{S}(\lambda) = \{\varphi \in AC([a,b],\mathbb{R}^n): \dot{\varphi}(t) \in F(t,\varphi(t),\lambda) \text{ a.e. in } [a,b],\ \varphi(a) = \theta_{\mathbb{R}^n}\}.$$

That theorem insures, in particular, that the multifunction $\lambda \to \mathcal{S}(\lambda)$ is lower semicontinuous, with respect to the topology (on $AC([a,b],\mathbb{R}^n)$) of uniform convergence. This readily implies that also the multifunction $\lambda \to \mathcal{A}(\lambda) = \{\varphi(b): \varphi \in \mathcal{S}(\lambda)\}$ is lower semicontinuous. Next, for every $\lambda \in \Lambda$, put

$$\mathfrak{S}(\lambda) = \{\varphi \in AC([a,b],\mathbb{R}^n): \exists\, u \in L^\infty([a,b],\mathbb{R}^m),\ \|u\|_{L^\infty([a,b],\mathbb{R}^m)} \le 1:$$
$$\dot{\varphi}(t) = A_\lambda(t)\,\varphi(t) + B_\lambda(t)\,u(t) \text{ a.e. in } [a,b],\ \varphi(a) = \theta_{\mathbb{R}^n}\},$$

where, of course, $A_\infty = A$, $B_\infty = B$.

Plainly, $\mathfrak{S}(\lambda) \subseteq \mathcal{S}(\lambda)$. On the other hand, if $\varphi \in \mathcal{S}(\lambda)$, we have $\dot{\varphi}(t) - A_\lambda(t)\,\varphi(t) \in B_\lambda(t)\,\Gamma$ a.e. in $[a,b]$, $\varphi(a) = \theta_{\mathbb{R}^n}$. Then, thanks to Theorem 7.1 of [4], there is a measurable function $u: [a,b] \to \Gamma$ such that $\dot{\varphi}(t) = A_\lambda(t)\,\varphi(t) + B_\lambda(t)\,u(t)$ a.e. in $[a,b]$. Hence $\varphi \in \mathfrak{S}(\lambda)$, that is $\mathcal{S}(\lambda) = \mathfrak{S}(\lambda)$. Now, consider the set $\mathcal{A}(\infty)$. Since the system (1) is completely controllable, one has $\operatorname{int}(\mathcal{A}(\infty)) \neq \emptyset$ (see Theorem 7.2.3 of [1]). Therefore, $\mathcal{A}(\infty)$ contains n linearly independent vectors $z_1, \dots, z_n$. Now, let $\delta > 0$ be such that if $\|v_i - z_i\|_{\mathbb{R}^n} < \delta$ $(i = 1, \dots, n)$, then the vectors $v_1, \dots, v_n$ are linearly independent (see, for instance, [2], Lemma 1). Recalling the lower semicontinuity of the multifunction $\lambda \to \mathcal{A}(\lambda)$, we then have that, for every sufficiently large $r \in \mathbb{N}$, the set $\mathcal{A}(k_r)$ meets, for each $i = 1, \dots, n$, the open ball around z_i of radius δ. Therefore, $\mathcal{A}(k_r)$, besides $\theta_{\mathbb{R}^n}$, contains n linearly independent vectors, and hence, since it is convex, its interior is non-empty. This implies (see again Theorem 7.2.3 and p. 97 of [1]) that the control system $\dot{x} = A_{k_r}(t)\,x + B_{k_r}(t)\,u$ is completely controllable, against one of the properties of the sequences $\{A_k\}$, $\{B_k\}$. This contradiction concludes the proof. ■

REMARK. We do not know, at present, if the conclusion of Theorem 1 holds when one replaces (3) with the following:

$$\int_a^b \frac{\|\widetilde{A}(t) - A(t)\|}{1 + \|\widetilde{A}(t) - A(t)\|}\, dt + \int_a^b \frac{\|\widetilde{B}(t) - B(t)\|}{1 + \|\widetilde{B}(t) - B(t)\|}\, dt < \rho\,.$$

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