Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the Aronszajn property for integral equations in Banach space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **83** (1989), n.1, p. 93–99.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_93_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1989.

Atti Acc. Lincei Rend. fis.
(8), LXXXIII (1989), pp. 93-99

Analisi funzionale. — *On the Aronszajn property for integral equations in Banach space.* Nota (*) di STANISŁAW SZUFLA, presentata dal Corrisp. R. CONTI.

ABSTRACT. — For the integral equation (1) below we prove the existence on an interval $J = [0, a]$ of a solution x with values in a Banach space E, belonging to the class $L^p(J, E)$, $p > 1$. Further, the set of solutions is shown to be a compact one in the sense of Aronszajn.

KEY WORDS: Integral equations; Banach spaces; Aronszajn property.

RIASSUNTO. — *Sulla proprietà di Aronszajn per le equazioni integrali negli spazi di Banach.* Usando il concetto di misura di non-compattezza si danno delle condizioni di compattezza per l'insieme di tutte le soluzioni L^p di un'equazione integrale non lineare di Volterra in uno spazio di Banach.

1. INTRODUCTION

Let $D = [0, d]$ be a compact interval in R and let E be a real Banach space. Denote by $L^p(D, E)$ $(p > 1)$ the space of all strongly measurable functions $u: D \to E$ with $\int_D \|u(t)\|^p \, dt < \infty$, provided with the norm $\|u\|_p = \left(\int_D \|u(t)\|^p \, dt \right)^{1/p}$.

In this paper we consider the integral equation

$$x(t) = g(t) + \int_0^t K(t, s) f(s, x(s)) \, ds. \tag{1}$$

We give sufficient conditions for the existence of a solution x of (1) belonging to the space $L^p(J, E)$, where $J = [0, a]$ is a subinterval of D. Moreover, we prove that the set S of all solutions $x \in L^p(J, E)$ of (1) is a compact R_δ in the sense of Aronszajn, i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts (cf. [1]). Our considerations are based on result of Browder and Gupta [2; Theorem 7] concerning topological properties of the set $T^{-1}(0)$ for a proper map T. Throughout this paper we shall assume that

1) $g \in L^p(D, E)$;

2) $(s, x) \to f(s, x)$ is a function from $D \times E$ into a Banach space H such that f is strongly measurable in s and continuous in x, and

$$\|f(s, x)\| \le c(s) + b\|x\|^{p/q} \qquad \text{for } s \in D \text{ and } x \in E,$$

where c is a nonnegative function belonging to $L^p(D, R)$, $b \ge 0$ and $q > 1$; let $r = q/(q-1)$.

3) K is a strongly measurable function from D^2 into the space of continuous linear mappings $H \to E$ such that $\|K(t, \cdot)\| \in L^r(D, R)$ for a.e. $t \in D$ and the function $t \to k(t) = \|K(t, .)\|_r$ belongs to $L^p(D, R)$.

(*) Pervenuta all'Accademia il 30 settembre 1987.

In contrast to the case $E = R^n$, the conditions 1)-3) are not sufficient for the existence of a solution of (1) when E is infinite dimensional, and therefore we must impose additional conditions on f. In Section 3 we shall show that the set of all L^p-solutions of (1) is compact R_δ whenever

$$\alpha(f(s,X)) \leq h(s)\,\alpha(X)$$

for $s \in D$ and for each bounded subset X of E, where α denotes the Kuratowski measure of noncompactness and $h \in L^m(D,R)$ for an $m > 1$. For example, the above condition holds if $f = f_1 + f_2$, where f_1 is completely continuous and

$$\|f_2(s,x) - f_2(s,y)\| \leq h(s)\|x - y\| \qquad (s \in D,\ x, y \in E).$$

In our proofs we use some ideas from the Mönch paper [6] concerning differential equations. Let us recall that in the last twenty years the measure of noncompactness has been employed for differential equations by many authors (e.g. Ambrosetti, Cellina, Deimling, Goebel, Lakshmikantham, Martin, Mönch, Pianigiani, Sadovskii, Szufla).

2. Measures of noncompactness

The Hausdorff measure of noncompactness β_Z in a Banach space Z is defined by

$$\beta_Z(X) = \inf\{\varepsilon > 0\colon\ X \text{ admits a finite } \varepsilon\text{—net in } Z\}$$

for any bounded subset X of Z. For properties of β_Z see [5, 8]. For convenience we shall denote by β and β_1 the Hausdorff measures of noncompactness in E and $L^1(D,E)$, respectively.

For any set V of functions from D into E we define a function v by $v(t) = \beta(V(t))$ $(t \in D)$, where $V(t) = \{x(t)\colon x \in V\}$ (under the convention that $\beta(X) = \infty$ if X is unbounded). The principal tool used in this work is the following theorem clarifying the relation between β and β_1.

Theorem 1 [7]. *Assume that the space E is separable and V is a countable set of functions belonging to $L^1(D,E)$. If there exists a function $\mu \in L^1(D,R)$ such that $\|x(t)\| \leq \mu(t)$ for all $x \in V$ and $t \in D$, then the corresponding function v is integrable and for any measurable subset T* of D

$$\beta\left(\left\{\int_T x(t)\,dt\colon x \in V\right\}\right) \leq \int_T v(t)\,dt.$$

Moreover, if $\lim\limits_{h\to 0} \sup\limits_{x\in V} \int_D \|x(t+h) - x(t)\|\,dt = 0$, *then*

$$\beta_1(V) \leq \int_D v(t)\,dt.$$

3. The main result

Assume, in addition to 1)-3), that

4) $p \geq q$; let m be such that $1/m + 1/r + 1/p = 1$ and $1 < m \leq \infty$; or

4′) $p \geq 2$ and $\|K\| \in L^p(D^2, R)$; let m be such that $1/m + 2/p = 1$ and $1 < m \leq \infty$.

THEOREM 2. *If g, f and K satisfy* 1)-3) *and* 4) *or* 4'), *and there exists a function $h \in L^m(D,R)$ such that*

$$\alpha(f(s,X)) \leq h(s)\,\alpha(X)$$

for $s \in D$ and for each bounded subset X of E, then there exists an interval $J=[0,a]$ such that the set S of all solutions $x \in L^p(J,E)$ of (1) *is a compact R_δ.*

PROOF. We choose a positive number $a < \min(d,\omega_+)$, where $[0,\omega_+)$ is the maximal interval of existence of the maximal absolutely continuous solution z_0 of the initial value problem

$$z' = 2^{p-1}(\|g(t)\| + k(t)\,\|c\|_q + bk(t)\,z^{1/q})^p\,, \qquad z(0)=0\,.$$

Let $J=[0,a]$, $L^p = L^p(J,E)$, $\rho^p = \max_{t \in J} z_0(t) + 1$ and $B = \{x \in L^p : \|x\|_p \leq \rho\}$. Put

$$F(x)(t) = \int_0^t K(t,s)\,f(s,x(s))\,ds \qquad \text{for } x \in L^p \text{ and } t \in J\,.$$

It is known that under the assumptions 2) and 3) F continuously maps L^p into itself and

$$\lim_{\tau \to 0} \sup_{x \in B} \int_0^a \|F(x)(t+\tau) - F(x)(t)\|\,dt = 0\,. \tag{3}$$

For any positive integer n and $x \in L^p$ put

$$F_n(x)(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a_n\,, \\ \int_0^{t-a_n} K(t,s)\,f(s,x(s))\,ds & \text{if } a_n \leq t \leq a\,, \end{cases}$$

where $a_n = a/n$. Then F_n is a continuous mapping $L^p \to L^p$. Moreover, 2), 3) and the Hölder inequality imply that

$$\|F_n(x)(t)\| \leq k(t)\,\|c\|_q + bk(t)\left(\int_0^t \|x(s)\|^p\,ds\right)^{1/q} \tag{4}$$

and

$$\|F(x)(t) - F_n(x)(t)\| \leq k_n(t)\left(\|c\|_q + b\left(\int_0^t \|x(s)\|^p\,ds\right)^{1/q}\right) \tag{5}$$

for $x \in L^p$, where

$$k_n(t) = \begin{cases} k(t) & \text{if } 0 \leq t \leq a_n\,, \\ \|K(t,\cdot)\,\chi_{[t-a_n,t]}\|_r & \text{if } a_n \leq t \leq a\,. \end{cases}$$

As $\lim_{n\to\infty} k_n(t) = 0$ and $k_n(t) \leq k(t)$ for a.e. $t \in J$, from (5) it follows that

$$\lim_{n\to\infty} \|F(x) - F_n(x)\|_p = 0 \qquad \text{uniformly in } x \in B\,.$$

Put $G(x) = g + F(x)$ and $G_n(x) = g + F_n(x)$ for $x \in B$. Then G and G_n are continuous mappings of B into L^p and

$$\lim_{n\to\infty} \|G(x) - G_n(x)\|_p = 0 \qquad \text{uniformly in } x \in B\,. \tag{6}$$

Fix n. It can be easily verified that for any $x, y \in B$

$$x - G_n(x) = y - G_n(y) \Rightarrow x = y. \tag{7}$$

Suppose that $x_j, x_0 \in B$ and

$$\lim_{j\to\infty} \|x_j - G_n(x_j) - x_0 + G_n(x_0)\|_p = 0. \tag{8}$$

Since $G_n(x_j)(t) = G_n(x_0)(t) = g(t)$ for $0 \le t \le a_n$, (8) implies that $\lim_{j\to\infty} \|(x_j - x_0) \cdot \chi_{[0,a_n]}\|_p = 0$. Further,

$$x_j(t) - x_0(t) = (x_j(t) - G_n(x_j)(t) - x_0(t) + G_n(x_0)(t)) + (F_n(x_j \chi_{[0,a_n]})(t) - F_n(x_0 \chi_{[0,a_n]})(t))$$

for $a_n \le t \le 2a_n$ and $j = 1, 2, \ldots$. By (8) and the continuity of F_n this proves that $\lim_{j\to\infty} \|(x_j - x_0)\chi_{[a_n, 2a_n]}\|_p = 0$. By repeating this argument we get $\lim_{j\to\infty} \|(x_j - x_0)\chi_{[0, ia_n]}\|_p = 0$ for $i = 1, 2, \ldots, n$, so that $\lim_{j\to\infty} \|x_j - x_0\|_p = 0$. From this and (7) it follows that the mapping $I - G_n: B \to L^p$ is a homeorphism into (I-the identity mapping).

We choose $\eta \in (0, 1/2)$ in such a way that the maximal continuous solution z_η of the integral equation

$$z(t) = \eta + 2^{p-1} \int_0^t (\|g(s)\| + k(s)\|c\|_q + bk(s)\, z^{1/q}(s))^p\, ds$$

is defined on J and $z_\eta(t) \le 1 + z_0(t)$ for $t \in J$.

Let $U = \{x \in L^p : \|x\|_p \le \eta\}$. For a given n and $y \in U$ we define a sequence of functions x_i, $i = 1, 2, \ldots, n$, by

$$\begin{aligned}
x_1(t) &= y(t) + g(t) && \text{for } 0 \le t \le a_n, \\
\tilde{x}_i(t) &= \begin{cases} x_i(t) & \text{for } 0 \le t \le i a_n, \\ 0 & \text{for } i a_n < t \le a, \end{cases} \\
x_{i+1}(t) &= x_i(t) && \text{for } 0 \le t \le i a_n, \\
x_{i+1}(t) &= y(t) + g(t) + F_n(\tilde{x}_i)(t) && \text{for } i a_n \le t \le (i+1) a_n.
\end{aligned}$$

Then $x_n \in L^p$ and $x_n(t) = y(t) + g(t) + F_n(x_n)(t)$ for $t \in J$.

In view of (4) we have

$$\|x_n(t)\| \le \|y(t)\| + \|g(t)\| + k(t)\|c\|_q + bk(t) \left(\int_0^t \|x_n(s)\|^p\, ds \right)^{1/q}$$

for $t \in J$. Putting $w_n(t) = \int_0^t \|x_n(s)\|^p\, ds$, we get

$$w_n(t) \le \|y\|_p + 2^{p-1} \int_0^t (\|g(s)\| + k(s)\|c\|_q + bk(s)\, w_n^{1/q}(s))^p\, ds \qquad \text{for } t \in J.$$

As $\|y\|_p \le \eta$, by the theorem on integral inequalities this implies that $w_n(t) \le z_\eta(t) \le z_0(t) + 1 \le \rho^p$ for $t \in J$. Thus $x_n \in B$.

This proves that

$$U \subset (I - G_n)(B) \qquad \text{for all } n. \tag{9}$$

Now we shall show that

(10) $(I-G)^{-1}(Y)$ is compact for each compact subset Y of L^p.

Let Y be a given compact subset of L^p and let (u_n) be a sequence in $(I-G)^{-1}(Y)$. Since $u_n - g - F(u_n) \in Y$ for $n = 1, 2, \ldots$, we can find a subsequence (u_{n_j}) and $y \in Y$ such that

(11) $$\lim_{j\to\infty} \|u_{n_j} - g - F(u_{n_j}) - y\|_p = 0 .$$

By passing to a subsequence if necessary, we may assume that

(12) $$\lim_{j\to\infty} (u_{n_j}(t) - g(t) - F(u_{n_j})(t)) = y(t) \qquad \text{for a.e. } t \in J .$$

Put $V = \{u_{n_j} : j = 1, 2, \ldots\}$ and $W = F(V)$. As $V \subset B$, from (3) and (11) it is clear that

(13) $$\lim_{\tau\to 0} \sup_{x\in W} \int_0^a \|x(t+\tau) - x(t)\|\, dt = 0$$

and

(14) $$\beta_1(V) = \beta_1(W) .$$

Since each strongly measurable function is a limit of an a.e. convergent sequence of simple functions, there exist a separable Banach subspace Z of E and a subset P_1 of J such that

$$\operatorname{mes}(J \setminus P_1) = 0 \text{ and } x(t) \in Z \qquad \text{for all } t \in P_1 \text{ and } x \in V \cup W .$$

On the other hand, (12) implies that there exists a subset P_2 of J such that $\operatorname{mes}(J \setminus P_2) = 0$ and

(15) $$\beta_Z(V(t)) = \beta_Z(W(t)) \qquad \text{for } t \in P_2 .$$

Let $P = P_1 \cap P_2$ and

$$v(t) = \begin{cases} \beta_Z(W(t)) & \text{for } t \in P \\ 0 & \text{for } t \in J \setminus P . \end{cases}$$

Since

(16) $$\|F(x)(t)\| \le k(t)\, \|c\|_q + bk(t)\, \rho^{p/q} \qquad \text{for } x \in B \text{ and } t \in J .$$

from (13) and Theorem 1 we deduce that the function v is integrable and

(17) $$\beta_1(W) \le \int_0^a v(t)\, dt .$$

Fix $t \in P$ such that $k(t) < \infty$. There exist a subset Q of P and a separable Banach subspace Z_t of E such that $\operatorname{mes}(J \setminus Q) = 0$ and $K(t,s) f(s, x(s)) \in Z_t$ for all $s \in Q$ and $x \in V$. Denote by T the closed linear hull of $Z \cup Z_t$. Obviously T is a separable Banach subspace of E.

Furthermore, by the Egoroff theorem and (12), for any $\varepsilon > 0$ there exists a closed subset J_ε of J such that $\operatorname{mes}(J \setminus J_\varepsilon) < \varepsilon$ and

$$\lim_{j\to\infty} (u_{n_j}(s) - g(s) - F(u_{n_j})(s)) = y(s) \text{ uniformly on } J_\varepsilon .$$

Hence, by the Luzin theorem, from this and (16) we infer that for a given $\varepsilon > 0$ there exist a closed subset A of $[0, t]$ and a positive number λ such that

$$\|u_{n_j}(s)\| \leq \lambda \qquad \text{for } s \in A \text{ and } j = 1, 2, \ldots;$$

the functions $s \to \|K(t,s)\|$ is continuous on A and

$$\|K(t,\cdot)\,\chi_M\|_r (\|c\|_q + b\rho^{p/q}) < \varepsilon, \tag{18}$$

where $M = [0, t] \setminus A$. Thus

$$\|K(t,s)\,f(s, x(s))\| \leq \mu(s) \qquad \text{fos } s \in A \text{ and } x \in V,$$

where $\mu(s) = \|K(t,s)\| (\|c\|_q + b\lambda^{p/q})$. Clearly, by 3°, the function μ is integrable on $[0, t]$.

Put

$$W_1(t) = \left\{ \int_A K(t,s)\,f(s, x(s))\, ds \colon x \in V \right\},$$

$$W_2(t) = \left\{ \int_M K(t,s)\,f(s, x(s))\, ds \colon x \in V \right\}.$$

Then

$$\begin{cases} W(t) \subset W_1(t) + W_2(t), \\ W_1(t) \subset \operatorname{mes} A \cdot \overline{\operatorname{conv}}\, \{K(t,s)\,f(s, x(s)) \colon x \in V,\, s \in A\} \subset T \end{cases} \tag{19}$$

and, similarly, $W_2(t) \subset T$.

By (2) we have

$$\beta_T(\{K(t,s)\,f(s,x(s)) \colon x \in V\}) \leq \alpha(\{K(t,s)\,f(s,x(s)) \colon x \in V\}) \leq$$
$$\leq \|K(t,s)\|\,\alpha(\{f(s,x(s)) \colon x \in V\}) \leq \|K(t,s)\|\,h(s)\,\alpha(V(s)) \leq$$
$$\leq 2\|K(t,s)\|\,h(s)\,\beta_Z(V(s)) \qquad \text{for } s \in A \cap Q.$$

Therefore, by Theorem 1,

$$\beta_T(W_1(t)) \leq \int_A \beta_T(\{K(t,s)\,f(s,x(s)) \colon x \in V\})\, ds \leq$$
$$\leq 2 \int_A \|K(t,s)\|\,h(s)\,\beta_Z(V(s))\, ds \leq 2 \int_0^t \|K(t,s)\|\,h(s)\,v(s)\, ds.$$

Moreover, as

$$\left\| \int_M K(t,s)\,f(s,x(s))\, ds \right\| \leq \|K(t,\cdot)\,\chi_M\|_r (\|c\|_q + b\rho^{p/q}) \qquad \text{for } x \in V,$$

(18) implies that $\beta_T(W_2(t)) \leq \varepsilon$. Hence, owing to (19),

$$v(t) = \beta_Z(W(t)) \leq 2\beta_T(W(t)) \leq 2\beta_T(W_1(t)) + 2\beta_T(W_2(t)) \leq 4 \int_0^t \|K(t,s)\|\,h(s)\,v(s)\, ds + 2\varepsilon.$$

As ε is arbitrary, we get

$$v(t) \le 4 \int_0^t \|K(t,s)\| h(s)\, v(s)\, ds. \tag{20}$$

By the Hölder inequality this imples

$$v(t) \le d(t)\, \|h\|_m \left(\int_0^t v^p(s)\, ds \right)^{1/p},$$

where

$$d(t) = \begin{cases} 4\|K(t,\cdot)\|_r & \text{if 4) holds}, \\ 4\|K(t,\cdot)\|_p & \text{if 4') holds}. \end{cases}$$

As $v \in L^p(J,R)$ and (20) holds for a.e. $t \in J$, putting $w(t) = \int_0^t v^p(s)\, ds$ we obtain

$$w'(t) \le d^p(t)\, \|h\|_m\, w(t) \qquad \text{for a.e. } t \in J$$

and $w(0) = 0$. From this we deduce that $w(t) = 0$ for $t \in J$. Consequently, by (17) and (14), $\beta_1(V) = 0$, so that V is relatively compact in L^1. Thus we can find a subsequence $(u_{n_{j_i}})$ of (u_{n_j}) which converges in L^1 to a function u_0. Moreover, (11) and (16) imply that the sequence (u_{n_j}) has equi-absolutely continuous norms in L^p. Hence the sequence $(u_{n_{j_i}})$ converges to u_0 in L^p. By (11) it is clear that $u_0 - G(u_0) = y \in Y$. This ends the proof of (10).

From (6), (9) and (10) it follows that the mapping $I - G$ satisfies all assumptions of Theorem 7 of [2]. Consequently, the set $(I-G)^{-1}(0)$ is a compact R_δ. On the other hand if $x \in S$, then analogously as for x_n in the proof of (9), it can be shown that $\|x\|_p \le \rho$, i.e. $x \in B$. Thus $S = (I-G)^{-1}(0)$ which ends the proof of Theorem 2.

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