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A remark on quasilinear elliptic variational inequalities with discontinuous coefficients

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Analisi matematica. — *A remark on quasilinear elliptic variational inequalities with discontinuous coefficients.* Nota (*) di Biagio Ricceri, presentata dal Socio G. Scorza Dragoni.

Abstract. — This Note contains the following remark on a recent result by Boccardo and Buttazzo: under the same assumptions, a stronger conclusion, concerning the solvability of variational inequalities, can be obtained.

Key words: Elliptic variational inequalities; Integral functionals; Pseudomonotone operators.

Riassunto. — *Un'osservazione sulle disequazioni variazionali ellittiche quasilineari a coefficienti discontinui.* Questa Nota contiene la seguente osservazione su un recente risultato di Boccardo e Buttazzo: nelle stesse ipotesi, è possibile ottenere una tesi più precisa, concernente la risolubilità di certe disequazioni variazionali.

Very recently, in [2], L. Boccardo and G. Buttazzo showed how, by using some new lower semicontinuity result for the functional of the Calculus of Variations (see [1]), it is possible to get existence theorems for a problem of the type

$$\begin{cases} -\sum_{i,j=1}^{n} D_i(a_{ij}(x,u)\,D_j u) = f & \text{in } \Omega \quad (f \in H^{-1}(\Omega)) \\ u \in H_0^1(\Omega) \end{cases}$$

when the coefficients $a_{ij}(x,s)$ are highly discontinuous in s.

Before recalling the existence result by Boccardo and Buttazzo, we fix some notation. Here and in the sequel, Ω is a non-empty bounded open subset of $\mathbb{R}^n$; $\mathcal{L}_n$ (resp. $\mathcal{L}_1$) is the Lebesgue σ-algebra of $\mathbb{R}^n$ (resp. of $\mathbb{R}$); if $E \in \mathcal{L}_n$, $m(E)$ denotes the Lebesgue measure of E; if $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, $\langle f, u\rangle$ denotes the value of f at u.

Theorem A ([2, Theorem 4.1]). *Let a_{ij} $(i,j = 1, \ldots, n)$ be n^2 real functions defined on $\Omega \times \mathbb{R}$ satisfying the following conditions:*

(1) *each function a_{ij} is $\mathcal{L}_n \otimes \mathcal{L}_1$-measurable and bounded;*

(2) *for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \Omega$, with $m(\Omega \setminus K_\varepsilon) < \varepsilon$, such that, for every $r > 0$, the functions of the family $\{a_{ij}(\cdot, s)\}_{|s| \leq r,\, i,j=1,\ldots,n}$ are equicontinuous on K_ε;*

(3) *there exists $\lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x,s)\,\xi_i \xi_j \geq \lambda \sum_{h=1}^{n} |\xi_h|^2$$

for almost every $x \in \Omega$ and every $s, \xi_1, \ldots, \xi_n \in \mathbb{R}$.

(*) Pervenuta all'Accademia il 13 ottobre 1989.

Then, for every $f \in H^{-1}(\Omega)$, *there exists* $u \in H_0^1(\Omega)$ *such that*

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x, u(x))\, D_j u(x)\, D_i v(x)\, dx = \langle f, v \rangle$$

for all $v \in H_0^1(\Omega)$.

The aim of this very short Note is simply to point out that, under the same assumptions of Theorem A, a stronger conclusion, concerning the solvability of variational inequalities, can be obtained.

Indeed, we have:

Theorem 1. *Let the assumptions of Theorem A be satisfied. Then, for every non-empty, closed, convex subset X of* $H_0^1(\Omega)$ *and for every* $f \in H^{-1}(\Omega)$, *there exists* $u \in X$ *such that*

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x, u(x))\, D_j u(x)\, D_i(u(x) - v(x))\, dx \leq \langle f, u - v \rangle$$

for all $v \in X$.

Proof. We will consider on $H_0^1(\Omega)$ the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^{n} \int_{\Omega} |D_i u(x)|^2\, dx \right)^{1/2}.$$

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator defined by putting

$$\langle Au, v \rangle = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x, u(x))\, D_j u(x)\, D_i v(x)\, dx$$

for all $u, v \in H_0^1(\Omega)$. From Theorem 2.1 of [2], we know that A is (well defined and) sequentially weakly continuous. Let $\mathcal{B}(\Omega)$ denote the Borel σ-algebra of Ω. For each $i, j = 1, \ldots, n$, by (1) and (2), taking into account, for instance, Theorem 6.1 of [4], it is easy to get a $\mathcal{B}(\Omega) \otimes \mathcal{L}_1$-measurable real function b_{ij} on $\Omega \times \mathbb{R}$ and a set $E_{ij} \in \mathcal{B}(\Omega)$, with $m(E_{ij}) = 0$, such that $a_{ij}(x, s) = b_{ij}(x, s)$ for all $(x, s) \in (\Omega \setminus E_{ij}) \times \mathbb{R}$. Therefore, for every $u \in H_0^1(\Omega)$, we have

$$\langle Au, u \rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(x, u(x))\, D_j u(x)\, D_i u(x)\, dx.$$

Consequently, by (3) and by Remark 4.7 of [1], the functional $u \to \langle Au, u \rangle$ is sequentially weakly lower semicontinuous. This fact together with the sequential weak continuity of A, implies at once that A is pseudomonotone (see [5, p. 217]). Next, if we put $M = \sup\{|a_{ij}(x, s)| : (x, s) \in \Omega \times \mathbb{R},\ i, j = 1, \ldots, n\}$, for every $u, v_0 \in H_0^1(\Omega)$, of course we have

$$\langle Au, u - v_0 \rangle \geq \lambda \|u\|^2_{H_0^1(\Omega)} - Mn^2 \|u\|_{H_0^1(\Omega)} \|v_0\|_{H_0^1(\Omega)}.$$

Now, the conclusion is, for instance, an immediate consequence of Theorem 4.17 of [5] (see also Proposition 3.1 and Remark 3.1, p. 43, of [3]). ■

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