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## Convergence results for periodic solutions of nonautonomous Hamiltonian systems

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**Analisi funzionale.** — *Convergence results for periodic solutions of nonautonomous Hamiltonian systems.* Nota di MARIO GIRARDI e MICHELE MATZEU, presentata (*) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — We prove some stability results for a certain class of periodic solutions of nonautonomous Hamiltonian systems in the case of Hamiltonian functions either with subquadratic growth or homogeneous with superquadratic growth. Thus we extend to the nonautonomous case some results recently established by the Authors for the autonomous case.

KEY WORDS: Convergence; Periodic solutions; Hamiltonian systems.

RIASSUNTO. — *Risultati di convergenza per soluzioni periodiche di sistemi Hamiltoniani non autonomi.* Si dimostrano alcuni risultati di stabilità per una certa classe di soluzioni di sistemi Hamiltoniani non autonomi nel caso di funzioni Hamiltoniane a crescita sottoquadratica, o a crescita superquadratica con ipotesi di omogeneità. Si estendono in tal modo al caso non autonomo alcuni risultati stabiliti di recente dagli Autori per il caso autonomo.

## INTRODUCTION

In [4] the authors proved some convergence results for periodic solutions of autonomous Hamiltonian systems with Hamiltonian functions $H$ having a subquadratic growth. In [5] analogous theorems were established in case that $H$ has a superquadratic behaviour, in the framework of assumptions of [1]. The aim of this paper is to extend these results to the case of *nonautonomous* Hamiltonian systems, that is to the case where $H$ depends on the time variable too. In this situation, the simple requirement of the pointwise convergence for Hamiltonian functions, which is sufficient for the autonomous case, is replaced by a more complicated assumption, which enables to apply an interesting result stated by Marcellini and Sbordone [6] in a quite different framework, for the study of $\Gamma$-convergence of convex integral functionals. We point out that, in a natural case where the required convergence is verified, the Hamiltonian system, obtained as the limit of a sequence of *nonautonomous* systems, is indeed *autonomous*.

Finally, we wish to thank V. Benci for stimulating discussions about the interest and the meaning of this kind of problems.

**1.** Let $H$: $R_+ \times R^{2N} \to R$ with $H(\cdot, z) \in C^0(R_+)$ $\forall z \in R^{2N}$, $H(t, \cdot) \in C^1(R^{2N})$ $\forall t \in R_+$ and such that

(1) $H(t, \cdot)$ is *strictly convex* on $R^{2N}$ $\forall t \in R_+$;

(2) $H(\cdot, z)$ is *periodic* on $R_+$, with *minimal period* $T > 0$, $\forall z \in R^{2N}$;

(*) Nella seduta del 13 maggio 1989.

(3) there exist three constant numbers $a_1$, $a_2 > 0$, $\alpha \in (1,2)$ such that

$$a_1|z|^\alpha \leq H(t,z) \leq a_2|z|^\alpha \qquad \forall z \in R^{2N},\ \forall t \in R_+ .$$

Let us consider, for every $n \in \mathbf{N}$, the following Hamiltonian system

$$(H_n) \qquad J\dot{z}_n = H'(nt, z_n(t)), \qquad T \text{ is the } \textit{minimal period} \text{ of } z_n$$

where, $\forall z = (x,y) \in R^N \times R^N$, $J(x,y) = (y, -x)$.

It is well known, that, under our assumptions on $H$, $(H_n)$ admits a solution $z_n$ which can be obtained through the duality principle by Clarke and Ekeland [2] ([1]). Precisely, let us consider the functional $I_n$ defined on the space

$$L_0^\beta = L_0^\beta(0,T;R^{2N}) = \left\{ v \in L^\beta(0,T;R^{2N}) \colon \int_0^T v(t)\,dt = 0 \right\} \quad (\beta = \alpha/(\alpha-1))$$

as

$$I_n(v) = \int_0^T G(nt, v(t))\,dt - \frac{1}{2}\int_0^T \langle \mathscr{L}^{-1} v(t), v(t)\rangle\,dt \qquad n \in \mathbf{N}$$

where $G$ is the Legendre transform of $H$ in the $z$-variable, *i.e.*

$$G(t,v) = \sup\{\langle z, v\rangle - H(t,z) \colon z \in R^{2N}\} \qquad \forall (t,v) \in R_+ \times R^{2N}$$

and $\mathscr{L}$ is the injective mapping $J \cdot d/dt$ defined from the space

$$H_{\neq}^{1,\beta} = \left\{ z \in H^{1,\beta}(0,T;R^{2N}) \colon z(0) = z(T),\ \int_0^T z(t)\,dt = 0 \right\}$$

into $L_0^\beta$. Then it is possible to show that $I_n$ admits a negative minimum on $L_0^\beta$, and that, if $u_n$ is a minimum point, then $z_n(t) = G'(nt, u_n(t))$ is in fact a solution of $(H_n)$. Let us call, from now on, a *T-minimum solution* of $(H_n)$ any solution $z_n$ of $(H_n)$ obtained in such a way.

Let us suppose now, that $\overline{H} \colon R_+ \times R^{2N} \to R$ is another function satisfying the same hypotheses as $H$.

One can state the following:

THEOREM 1. *Let* $H_n(t,z) \overset{\text{def}}{=} H(nt,z)$, $\forall (t,z) \in R_+ \times R^{2N}$, $\forall n \in \mathbf{N}$, *and let* $H_n(\cdot, z)$ *converge to* $\overline{H}(\cdot, z)$ *in the weak* $*$*-topology of* $L^\infty(0,T)$ *for any* $z \in R^{2N}$. *Let us consider the problem*

$$(\overline{H}) \qquad J\dot{z}(t) = \overline{H}'(t, z(t)), \qquad T \text{ is the } \textit{minimal period} \text{ of } z.$$

*Then one has*

(A1) *If* $\{z_n\}$ *is a sequence converging to* $z$ *in* $H_{\neq}^{1,\beta}$ *with* $z_n$ *T-minimum solutions of* $(H_n)$ $\forall n \in \mathbf{N}$, *then* $z$ *is a T-minimum solution of* $(\overline{H})$.

([1]) Actually, in [2], the Hamiltonian function $H$ only depends on $z$, but it is easy to check that the arguments given in [2] still hold in the present case $H = H(t,z)$.

(A2) *Every sequence* $\{z_n\}$, *with* $z_n$ *T-minimum solution of* $(H_n)$ $\forall n \in \mathbb{N}$, *admits a subsequence weakly converging in* $H^{1,\beta}_{\neq}$ *to a T-minimum solution* $z$ *of* $(\overline{H})$.

(A3) *The following alternative holds*:

$(A3)_1$ *Any sequence* $\{z_n\}$ *with* $z_n$ *T-minimum solution of* $(H_n)$, *converges weakly to a T-minimum solution* $z$ *of* $(\overline{H})$ *in the quotient space* $H^{1,\beta}_{\neq}/\sim$, *where* $\sim$ *is the equivalence defined as*

$$\forall z_1, z_2,\ z_1 \sim z_2 \quad \textit{iff}\ z_1(t+s) = z_2(t)\ \forall t \in [0,T],\ \textit{for some}\ s \in [0,T].$$

$(A3)_2$ *There exist at least* 2 *T-minimum solutions of* $(\overline{H})$.

Remark 1. Conditions (1), (3) are automatically verified by $\overline{H}$, when one requires the convergence assumption of Theorem 1.

Remark 2. The assumption $\overline{H}(t,\cdot) \in C^1(R^{2N})$ $\forall t \in R_+$ can be indeed weakened by the assumption $\overline{H}(t,\cdot) \in C^0(R^{2N})$ $\forall t \in R_+$. In such a case the Hamiltonian system $(\overline{H})$ must be interpreted in a weak sense, that is the *differential equation* in $(\overline{H})$ must be replaced by a *differential inclusion*, and $(\overline{H})$ becomes

$$(\overline{H}_\partial) \qquad J\dot{z}(t) \in \partial\overline{H}(t, z(t)), \qquad T \text{ is the } \textit{minimal period} \text{ of } z,$$

where $\partial\overline{H}$ denotes the *subdifferential* of $\overline{H}$ in the $z$-variable, that is the *set* defined as

$$\overline{H}(t,z) = \{v \in R^{2N} : \overline{H}(t,w) \geqslant \overline{H}(t,z) + \langle v, z-w\rangle\ \forall w \in R^{2N}\}.$$

Of course, in this (more general) case too, one can define the concept of $T$-minimum solution for $(\overline{H}_\partial)$, in the sense that, in analogy with the case $\overline{H}(t,\cdot) \in C^1(R^{2N})$, one can easily prove that any minimum point $u$ of the functional

$$\overline{I}(v) = \int_0^T \overline{G}(t, v(t))\, dt - \frac{1}{2}\int_0^T \langle \mathscr{L}^{-1}v(t), v(t)\rangle\, dt \qquad \forall v \in L^\beta_0$$

($\overline{G}$ being the Legendre transform of $\overline{H}$) is associated with a solution $z = \mathscr{L}^{-1}u$ of $(\overline{H}_\partial)$, and still one can state Theorem 1.

Corollary 1. *Let* $H(t,z) = H_0(z)\,\varphi(t)$, *where* $H_0 \in C^1(R^{2N})$, $\varphi \in C^0(R_+)$ *and*

(4) $H_0$ *is* strictly convex *on* $R^{2N}$;

(5) $a_1|z|^\alpha \leqslant H_0(z) \leqslant a_2|z|^\alpha$ $\forall z \in R^{2N}$, $a_1, a_2 > 0$, $\alpha \in (1,2)$;

(6) $\varphi$ *is periodic on* $R_+$, *with minimal period* $T$;

(7) $\exists c > 0$: $\varphi(t) \geqslant c > 0$ $\forall t \in R_+$.

*Let us consider the following autonomous Hamiltonian system*

$$(H_0) \qquad J\dot{z}(t) = H_0'(z(t))\,\varphi_0, \qquad T \textit{ is the } \text{minimal period } \textit{of } z,$$

*where* $\varphi_0 = (1/T)\int_0^T \varphi(t)\,dt$. *Then* (A1), (A2), (A3) *hold with* $H(t,z) = \overline{H}(z) = H_0(z)\,\varphi_0$ *and* $(\overline{H}) = (H_0)$.

REMARK 3. Let us note that, by Corollary 1, the sequence of *nonautonomous* Hamiltonian systems $(H_n)$ «converges» (in the sense of $T$-minimum solutions) to an *autonomous* Hamiltonian system, such as $(H_0)$.

PROOF OF COROLLARY 1. It is enough to observe that the sequence $H_n(t,z) = H(nt,z) = H_0(z)\,\varphi(nt)$ converges in the weak $*$-topology of $L^\infty(0,T)$ to $H_0(z)\,\varphi_0$, for any $z \in R^{2N}$.

For the proof of Theorem 1, one uses the following basic

LEMMA 1. *Under the assumptions of Theorem 1, the sequence of functionals* $\{I_n\}$ *$\Gamma$-converges to the functional* $\bar{I}$ *in* $L_0^\beta$*-weak, that is* (*see* [3] e.g.)

$$(\Gamma_1) \qquad \forall \bar{v} \in L_0^\beta, \exists \{v_n\} \subset L_0^\beta, \qquad \textit{with } v_n \xrightarrow{L^\beta} \bar{v}, \text{ s.t. } I_n(v_n) \to \bar{I}(\bar{v}),$$

$$(\Gamma_2) \qquad \forall \bar{v} \in L_0^\beta, \forall \{v_n\} \subset L_0^\beta, \qquad \textit{with } v_n \xrightarrow{L^\beta} \bar{v}, \Rightarrow \bar{I}(\bar{v}) \leq \underline{\lim}\, I_n(v_n).$$

PROOF. One can verify that it is possible to apply the following general result due to Marcellini and Sbordone, in the framework of $\Gamma$-convergence of convex integral functionals:

PROPOSITION 1 ([6, Thm. 3.4.]). *Let* $f_n(x,z)$ *and* $f(x,z)$ *be measurable functions in* $x \in \Omega$ *(a bounded open subset of* $R^k$*), convex in* $z \in R^h$ (2) *and such that* $\lambda|z|^p \leq f_n(x,z)$, $\lambda|z|^p \leq f(x,z)$, *for some* $\lambda > 0$, $p > 1$, $f_n(x,0) = f(x,0) = 0$, $\forall x \in \Omega$, $\forall z \in R^h$. *Putting, on* $L^p(\Omega; R^h)$,

$$F_n(v) = \int_\Omega f_n(x, v(x))\,dx, \qquad F(v) = \int_\Omega f(x, v(x))\,dx$$

*then* $\{F_n\}$ *$\Gamma$-converges to* $F$ *in* $L^p(\Omega; R^h)$-weak, *if and only if* $\{f_n^*(\cdot,z)\}$ *converges to* $f^*(\cdot,z)$ *in the weak $*$-topology of* $L^\infty(\Omega; R^h)$ $\forall z \in R^h$, *where* $f_n^*(\cdot,z)$ *and* $f^*(\cdot,z)$ *are the Legendre transforms of* $f_n$ *and* $f$ *in* $z$.

In our present case, one chooses $k = 1$, $\Omega = (0,T)$, $h = 2N$, $f_n(x,z) = f_n(t,z) = G(nt,z)$, $f(x,z) = f(t,z) = \overline{G}(t,z)$, $\lambda = a_1$, $p = \beta$, $f_n^* = H_n$. So Proposition 1 yields the $\Gamma$-convergence in $L^\beta$-weak of the functionals

$$F_n(v) = \int_0^T \overline{G}(nt, v(t))\,dt$$

to the functional

$$F(v) = \int_0^T \overline{G}(t, v(t))\,dt.$$

Finally, the compactness property of $\mathscr{L}^{-1}$ gives the $\Gamma$-convergence of $\{I_n\}$ to $\bar{I}$ in $L^\beta$-weak and the $\Gamma$-convergence in $L_0^\beta$-weak, as a consequence of the fact that $L_0^\beta$ is closed in $L^\beta$.

(2) Actually the result is proved with $k = h$ in [6], but it still holds for any arbitrary choice of $k$ and $h$ in N, as one can verify.

PROOF OF THEOREM 1. Lemma 1 permits to use arguments which are analogous to the ones given in the proof of Thm. 1 of [4]. Hence the proof of the actual theorem follows.

**2.** Let now $H(t,z) = H_0(z)\,\varphi(t)$, where $H_0 \in C^1(R^{2N})$, $\varphi \in C^0(R_+)$ and

(8) $H_0$ is *strictly convex* on $R^{2N}$;

(9) $H_0$ is *positively homogeneous* with degree $\beta > 2$, that is

$$H_0(\lambda z) = |\lambda|^\beta H_0(z) \qquad \forall z \in R^{2N},\ \forall \lambda \in R;$$

(10) $\varphi$ is periodic on $R_+$, with *minimal period* $T$;

(11) $\exists c > 0$: $\varphi(t) \geq c > 0$ $\forall t \in R_+$.

Putting $H_n(t,z) = H_0(z)\,\varphi(nt)$, $\forall n \in \mathrm{N}$, $\forall (t,z) \in R_+ \times R^{2N}$, conditions (8) ÷ (11) enable us to apply a result due to Ambrosetti and Mancini [1] (3) to the Hamiltonian system

$$(H_n) \qquad J\dot{z}_n(t) = H_n'(t, z_n(t)) = H_0'(z_n(t))\,\varphi(nt), \qquad T \text{ is the minimal period of } z_n$$

in order to state that, for any $n \in \mathrm{N}$, $(H_n)$ admits a solution $z_n$ obtained as $z_n = \mathscr{L}^{-1} u_n$ where $\mathscr{L}$ is the injective mapping $J \cdot d/dt$ from the space

$$H^{1,\alpha}_{\neq} = \left\{ z \in H^{1,\alpha}(0,T;R^{2N}) \colon z(0) = z(T),\ \int_0^T z(t)\,dt = 0 \right\} \qquad (\alpha = \beta/(\beta-1))$$

into the space

$$L^\alpha_0 = \left\{ v \in L^\alpha(0,T;R^{2N}) \colon \int_0^T v(t)\,dt = 0 \right\}$$

and $u_n$ is a minimum point of the functional

$$I_n(v) = \int_0^T G(nt, v(t))\,dt - \frac{1}{2}\int_0^T \langle \mathscr{L}^{-1} v(t), v(t) \rangle\,dt \qquad \forall v \in L^\alpha_0$$

($G$ being the Legendre transform of $H$ in the $z$-variable), on the smooth manifold

$$M_n = \left\{ v \in L^\alpha_0 \setminus \{0\} \colon \int_0^T \langle \mathscr{L}^{-1} v(t), v(t) \rangle\,dt = \int_0^T \langle G'(nt, v(t)), v(t) \rangle\,dt \right\}.$$

Analogously as in [1], let us call a *T-minimum solution* of $(H_n)$ any solution $z_n$ obtained in such a way.

Putting, as in § 1,

$$\varphi_0 = (1/T) \int_0^T \varphi(t)\,dt$$

(3) The same remark in (1), related to the result by Clarke and Ekeland [2], must be done in this case too.

let us consider the following *autonomous* Hamiltonian system

$(H_0)$ $$J\dot z(t) = H'(z(t))\,\varphi_0\,, \qquad T \text{ is the } \textit{minimal period} \text{ of } z.$$

Of course, $(H_0)$ too, as $(H_n)$, admits a $T$-minimum solution related to the minimization of the functional

$$I_0(v) = \int_0^T G_0(v(t))\,dt - \frac{1}{2}\int_0^T \langle \mathcal{L}^{-1}v(t), v(t)\rangle\,dt \qquad \forall v \in L_0^\alpha \tag{12}$$

($G_0(\cdot)$ being the Legendre transform of $H_0(\cdot)\,\varphi_0$), on the smooth manifold

$$M_0 = \left\{ v \in L_0^\alpha \setminus \{0\} : \int_0^T \langle \mathcal{L}^{-1}v(t), v(t)\rangle\,dt = \int_0^T \langle G_0'(v(t), v(t))\rangle\,dt \right\}. \tag{13}$$

As for the convergence of $T$-minimum solutions of $(H_n)$ to $T$-minimum solutions of $(H_0)$, one can state an analogous result as that of §1, given by the following

THEOREM 2. *Let* (8) $\div$ (11) *be assumed. Then one has*:

(A1) *If* $\{z_n\}$ *is a sequence weakly converging to* $z$ *in* $H_{\neq}^{1,\alpha}$, *with* $z_n$ *T-minimum solution of* $(H_n)$ $\forall n \in \mathbb{N}$, *then* $z$ *is a T-minimum solution of* $(H_0)$.

(A2) *Every sequence* $\{z_n\}$, *with* $z_n$ *T-minimum solution of* $(H_n)$ $\forall n \in \mathbb{N}$, *admits a subsequence weakly converging in* $H_{\neq}^{1,\alpha}$ *to a T-minimum solution of* $(H_0)$.

(A3) *The following alternative holds*:

$(A3)_1$ *Any sequence* $\{z_n\}$, *with* $z_n$ *T-minimum solution of* $(H_n)$ *converges weakly to a T-minimum solution* $z$ *of* $(H_0)$, *in the quotient space* $H_{\neq}^{1,\alpha}/\sim$, *where* $\sim$ *is the equivalence defined as*

$$\forall z_1, z_2,\ z_1 \sim z_2 \text{ iff } z_1(t+s) = z_2(t)\ \forall t \in [0,T] \textit{ for some } s \in [0,T].$$

$(A3)_2$ *There exist at least* 2 *T-minimum solutions of* $(H_0)$.

PROOF OF THEOREM 2. First of all, let us prove that, if $z_n$ is a $T$-minimum solution of $(H_n)$ $\forall n \in \mathbb{N}$, and $z_n \rightharpoonup z$ in $H_{\neq}^{1,\alpha}$, then $z$ solves the differential equation in $(H_0)$, so $u = \mathcal{L}z$ belongs to $M_0$. Indeed, for any $v \in L^1(0,T;R^{2N})$, one has

$$\int_0^T \langle H_0'(z_n(t))\,\varphi(nt), v(t)\rangle\,dt \to \int_0^T \langle H_0'(z(t))\,\varphi_0, v(t)\rangle\,dt\,,$$

and, since $J\dot z_n \rightharpoonup J\dot z$ then $z$ solves the differential equation in $(H_0)$. At this point, in order to state (A1), it is enough to prove the following statement:

$$\forall \bar v \in M_0, \exists v_n \in M_n \ \textit{s.t.}\ v_n \rightharpoonup \bar v \quad \text{in } L_0^\alpha \quad \text{and} \quad I_n(v_n) \to I_0(\bar v). \tag{14}$$

Let us prove (14). Let $w_n \in L_0^\alpha$ be such that $w_n \rightharpoonup \bar v$ in $L_0^\alpha$ and

$$\int_0^T G(nt, w_n(t))\,dt \to \int_0^T G_0(\bar v(t))\,dt:$$

the existence of such a sequence $\{w_n\}$ in ensured by the $\Gamma$-convergence of the functionals

$$\int_0^T G(nt, v(t))\,dt$$

to the functional

$$\int_0^T G_0(v(t))\,dt$$

in the weak topology of $L_0^\alpha$ (which can be proved by the same arguments given in the proof of Lemma 1 of § 1).

Let now $\{r_n\} \subset R$ be such that $v_n = r_n w_n$ belongs to $M_n$ (for the proof of the existence and uniqueness of such a number $r_n$, in a more general framework, see [1]). The sequence $\{r_n\}$ is bounded in $R$ and $|r_n| \geqslant \text{const} > 0$: in fact the $\beta$-homogeneity of $H_0$ (so the $\alpha$-homogeneity of $G_0$ as well) implies that, as $v_n \in M_n$, there exist some numbers $d_1$, $d_2 > 0$ such that:

$$d_1|r_n|^\alpha \int_0^T |w_n(t)|^\alpha\,dt \leqslant r_n^2 \int_0^T \langle \mathscr{L}^{-1} w_n(t), w_n(t)\rangle\,dt \leqslant d_2|r_n|^\alpha \int_0^T |w_n(t)|^\alpha\,dt. \tag{15}$$

On the other side, $\{w_n\}$ is bounded in $L_0^\alpha$ and $\|w_n\| \geqslant \text{const} > 0$, as $w_n \rightharpoonup \overline{v} \neq 0$. Since $\left\{\int_0^T \langle \mathscr{L}^{-1} w_n(t), w_n(t)\rangle\,dt\right\}$ converges to $\int_0^T \langle \mathscr{L}^{-1}\overline{v}(t), \overline{v}(t)\rangle\,dt = \int_0^T \langle G_0'(\overline{v}(t)), \overline{v}(t)\rangle\,dt$, which is different from 0, it follows, as $\alpha < 2$, that the first inequality in (15) implies $|r_n| \geqslant \text{const} > 0$, the second one implies the boundedness of $\{r_n\}$.

Let now $\{r_{n_j}\}$ be a subsequence of $\{r_n\}$ converging to some $\overline{r} \in R\setminus\{0\}$, so $v_{n_j} = r_{n_j} w_{n_j}$ weakly converges to $\tilde{v} = \overline{r}\,\overline{v} \neq 0$. It remains to prove that

$$\overline{r} = 1 \qquad (\text{so } v_n = r_n w_n \rightharpoonup \tilde{v} = \overline{v}), \tag{16}$$

$$\int_0^T G(nt, v_n(t))\,dt \to \int_0^T G_0(\overline{v}(t))\,dt. \tag{17}$$

Observe that (17) implies that $I_n(v_n) \to I_0(\overline{v})$, as a consequence of the compactness property of $\mathscr{L}^{-1}$.

As for (16), one has to show that $\tilde{v}$ belongs to $M_0$. Indeed, since $w_n \in M_n$, one has

$$\int_0^T \langle \mathscr{L}^{-1} v_{n_j}(t), v_{n_j}(t)\rangle\,dt = r_{n_j}^2 \int_0^T \langle \mathscr{L}^{-1} w_{n_j}(t), w_{n_j}(t)\rangle\,dt =$$

$$= \int_0^T \langle G'(n_j t, r_{n_j} w_{n_j}(t)), r_{n_j} w_{n_j}(t)\rangle\,dt = \alpha \int_0^T G(n_j t, r_{n_j} w_{n_j}(t))\,dt = \alpha r_{n_j}^\alpha \int_0^T G(n_j t, w_{n_j}(t))\,dt.$$

So, by passing to the limit as $j \to +\infty$, one gets

$$\int_0^T \langle \mathscr{L}^{-1}\tilde{v}(t), \tilde{v}(t)\rangle\,dt = \alpha \overline{r}^\alpha \int_0^T G_0(\overline{v}(t))\,dt = \alpha \int_0^T G_0(\overline{r}\,\overline{v}(t))\,dt = \int_0^T \langle G_0'(v(t)), v(t)\rangle\,dt,$$

therefore, as $\tilde{v} \neq 0$, $\tilde{v}$ belongs to $M_0$, and (16) follows.

Let us now prove (17). One has

$$\int_0^T G(nt, v_n(t))\, dt = r_n^\alpha \int_0^T G(nt, w_n(t))\, dt \to \int_0^T G_0(\overline{v}(t))\, dt$$

and (17) is proved.

The statement (A2) can be deduced from the compactness property of $\mathscr{L}^{-1}$ and by proving the boundedness, in $L_0^\alpha$, of the set

$$\mathscr{M} = \bigcup_{n \in \mathrm{N}} \{w \in M_n : I_n(w) \leq I_n(v) \ \ \forall v \in M_n\}\,.$$

[Indeed, if $w \in \mathscr{M}$ and $w$ minimizes $I_n$ on $M_n$, then it easily follows that $I_n(w) > 0$ and $\int_0^T \langle \mathscr{L}^{-1} w(t), w(t)\rangle\, dt > 0$, so that, for some $d_1$, $d_2 > 0$,

$$0 < I_n(w) d_1 \|w_n\|^\alpha_{L^\alpha} - d_2 \|w_n\|^2_{L^\alpha}\,,$$

which implies the boundedness of $\mathscr{M}$, as $\alpha < 2$].

Finally (A3) follows from (A1) and (A2) by an obvious argument.

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