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An integrality criterion for elliptic modular forms

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Teoria dei numeri. — *An integrality criterion for elliptic modular forms.* Nota (*) di ANDREA MORI, presentata dal Corrisp. C. PROCESI.

ABSTRACT. — Let f be an elliptic modular form level of N. We present a criterion for the integrality of f at primes not dividing N. The result is in terms of the values at CM points of the forms obtained applying to f the iterates of the Maaß differential operators.

KEY WORDS: Modular forms; Modular curves; Complex multiplications.

RIASSUNTO. — *Un criterio di integralità per forme modulari ellittiche.* Si enuncia un criterio di integralità per i primi non dividenti il livello per forme modulari ellittiche. Il criterio si basa sui valori assunti in certi punti particolari del semipiano a parte immaginaria positiva dalle forme ottenute applicando gli iterati degli operatori di Maaß alla forma in esame.

1. MOTIVATIONS

Let $\mathcal{H} = \{z = x + iy \in \mathbf{C} \mid y > 0\}$. The group $SL_2(\boldsymbol{R})$ acts on $\mathcal{H}$ via linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)(cz + d)^{-1}.$$

Let Γ be a subgroup of finite index in the full modular group $SL_2(\boldsymbol{Z})$. The action of $SL_2(\boldsymbol{R})$ on $\mathcal{H}$ extends to an action of Γ on $\mathcal{H} \cup \boldsymbol{P}^1(\mathbf{Q})$ and the orbits in $\boldsymbol{P}^1(\mathbf{Q})$ are called the *cusps* of Γ.

Classically, one gives the

(1.1) DEFINITION. An elliptic modular form f of weight k, ($k \in \boldsymbol{Z}$) relative to Γ is an holomorphic function on $\mathcal{H}$ such that

a) $f(\gamma z) = (cz + d)^k f(z)$, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

b) f is holomorphic at the cusps.

Moreover, f is called a cusp form if it is 0 at the cusps.

The meaning of condition *b*) is roughly the following: because of condition *a*), f is forced to be periodic of period N, for some $N \in \boldsymbol{Z}$. Then, for each cusp $s \in \boldsymbol{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ we can map $\mathcal{H}$ holomorphically onto the punctured disc $D^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ in such a way that $s \mapsto 0$ and f descends to a function f_s on D^*. Then the requirement is that each f_s must be holomorphically prolongable to the whole disc $D = D^* \cup \{0\}$. For a more detailed and precise discussion see [13, §2.1].

The resulting power series are called the *Fourier expansions* of f at the different cusps.

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(1.2) DEFINITION. When $\Gamma=\Gamma(N)=\{\gamma\in SL_2(\mathbf{Z})\mid\gamma\equiv 1 \bmod N\}$, f is called an elliptic holomorphic modular form of level N (and weight k).

Although some of the following considerations hold for elliptic modular forms with respect to any Γ as above, we will limit ourselves to the N-level case. Let us denote $G_k(N)$ the space of elliptic holomorphic modular forms of level N and weight k. It is well-known that the spaces $G_k(N)$ are finite dimensional and in fact trivial for $k\leqslant 0$, see [13, §2.6].

Ramanujan was the first to observe that the Fourier coefficients of elliptic holomorphic modular forms satisfy congruence properties: analyzing the function

$$\Delta(q)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)\,q^n\,,\qquad q=e^{2\pi i z}\,,$$

the unique cusp form of level 1 and weight 12, he discovered that $\tau(n)\equiv$ $\equiv\sigma_{11}(n)$ mod 691, where

$$\sigma_t(n)=\sum_{d|n}d^t.$$

Starting with the work [10], a great deal of different congruences for the coefficients $\tau(n)$ have been discovered using classical methods; see the introduction of [14] for a list and the relative references.

In order to provide a unitary theory of such congruence properties, Serre initiated in the late 60's the modern theory of modular forms introducing, in a purely algebraic way, his modular forms modulo primes and relating the Fourier coefficients of elliptic holomorphic modular forms to certain l-adic representations, see [11, 12].

The next step was made by Deligne and Katz: reconsidering the earlier work [3] they wanted to reinterpret Serre's work in terms of the arithmetic of the space of moduli of elliptic curves with level structure (the so-called modular curves $Y(N)$, whose complex points are given by the quotient $\Gamma(N)\diagdown\mathcal{H}$).

To start the theory one needs a general algebraic theory of modular forms, *i.e.* one would like to work over any base ring R_0, the classical definition 1.1 corresponding to the case $R_0=\mathbf{C}$. The definition is as follows:

(1.3) DEFINITION ([4]). Let R_0 be any ring. A modular form of weight k and level N defined over R_0 is a function f of triples $(E,\,\omega,\,\alpha_N)$, where E is an elliptic curve defined over an R_0-algebra R, ω is an R-rational invariant 1-form on E and α_N is an N-level structure on E, such that

a) $f(E,\,\omega,\,\alpha_N)\in R$;

b) $f(E,\,\lambda\omega,\,\alpha_N)=\lambda^{-k}f(E,\,\omega,\,\alpha_N)$ for each $\lambda\in R^{\times}$;

c) $f(E,\,\omega,\,\alpha_N)$ depends only on the R-isomorphism class of $(E,\,\omega,\,\alpha_N)$;

d) for any map ψ: $R\to S$ of R_0-algebras, $f(E\otimes S,\,\psi^*\omega,\alpha_N)=\psi(f(E,\omega,\alpha_N))$.

This definition 1.3 extends definition 1.1 in the following sense: an elliptic holomorphic modular form f gives rise to a modular form f_{alg} defined over $\mathbf{C}$ by the formula

$$f_{\mathrm{alg}}(\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau, dz) = f(\tau) \tag{1}$$

for $\tau \in \mathcal{H}$, and making $f_{\mathrm{alg}}(E/\mathbf{C}, \omega)$ compatible with (1) and definition 1.3*b*).

2. A general problem. The q-expansion principle

It is clear from definition 1.3 that if $R_0 \subset S_0$ is an inclusion of rings, modular forms defined over R_0 give rise to modular forms (of the same level and weight) defined over S_0. It is very natural to pose the following general

(2.1) Problem. Let $R_0 \subset S_0$ be two rings and let f be a modular form defined over S_0. Under what conditions is f in fact defined over R_0?

A satisfactory solution to this problem can be given when the level N is invertible in R_0 and R_0 also contains a primitive N-th root of unity ζ_N. If this is the case, one can define the Fourier expansions (also called *q-expansions*) as the values of f at the triples (Tate $(q^N) \otimes R_0, \omega_{\mathrm{can}}, \alpha_N$) where α_N runs through the different N-level structures of Tate (q^N). Thus $f(\text{Tate } (q^N) \otimes R_0, \omega_{\mathrm{can}}, \alpha_N) \in \mathbf{Z}[1/N, \zeta_N]((q)) \otimes R_0$, and f is said *holomorphic* if its q-expansions belong in fact to the ring $\mathbf{Z}[1/N, \zeta_N][[q]] \otimes R_0$.

The following result holds:

(2.2) Theorem (*q-expansion principle*, [4, § 1.6.1-2]). Let $N \geqslant 3$, R_0 and S_0 two $\mathbf{Z}[1/N]$-algebras with $\zeta_N \in R_0 \subset S_0$, and f a holomorphic modular form of level N and weight k defined over S_0. Suppose that on each of the connected components of $Y(N) \underset{\mathbf{Z}[1/N]}{\otimes} \mathbf{Z}[1/N, \zeta_N]$ there is at least one cusp at which the coefficients of the q-expansion of f belong to $R_0 \underset{\mathbf{Z}[1/N]}{\otimes} \mathbf{Z}[1/N, \zeta_N]$. Then f is defined over R_0.

Observations:

a) Theorem 2.2 can be extended to levels 1 and 2 if we look at modular forms of level 1 and 2 as particular modular forms of higher level, see [4, §1.9-10]. The difficulty comes from the fact that the functor «isomorphism classes of elliptic curves with N-level structure» is not representable for $N = 1, 2$ due to the existence of non-trivial automorphisms.

b) Similar results hold for modular forms in several variables (Siegel modular forms), see [2].

3. An integrality criterion

We shall now fix our attention on a special case of the problem 2.1; namely, we consider the case $S_0 = \mathbf{C}$, f an elliptic holomorphic modular form and $R_0 = \mathcal{O}_{(v)}$, the valuation ring associated to a non-archimedean place v in some given number field K. Let p be the rational prime lying under the maximal ideal $\mathfrak{p}_v \subset \mathcal{O}_{(v)}$. We shall assume that p does not divide the level N, so that N is invertible in $\mathcal{O}_{(v)}$.

As shown by the author in his thesis [9], problem 2.1 can be solved in the special setting described above, looking at the values of f and the C^∞-functions obtained from f applying the iterates of the Maaß differential operators δ_k (to be described briefly in §4) at special points of $\mathcal{H}$. The precise statement (a complete proof of which will appear in a forthcoming paper) is as follows:

(3.1) THEOREM (*Integrality criterion*). Let f be an elliptic holomorphic modular form of weight k and level $N \geq 3$. Let K be a number field with $\zeta_N \in K$, v a non-archimedean place of K not dividing N, and $\mathcal{O}_{(v)} \subset K$ its valuation ring. Let E be an elliptic curve defined over K with K-rational complex multiplications and with ordinary good reduction modulo v, such that its N-torsion is K-rational. Let $\tau \in \mathcal{H}$ be such that $E \otimes \mathbf{C} \simeq \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$. Then f is defined over $\mathcal{O}_{(v)}$ if and only if the numbers

$$c_r(f) = ((-4\pi)^r/\Omega_E^{k+2r})(\delta_k^{(r)} f)(\tau)$$

belong to $\mathcal{O}_{(v)}$ and

$$v\left(\sum_{i=1}^{r} b_{i,r} c_i(f)\right) \geq v(r!) \tag{2}$$

for $r = 0, 1, \ldots$, where $\delta_k^{(r)}$ is the r-th iterate of δ_k, Ω_E is the period of E and

$$\sum_{i=0}^{r} b_{i,r} X^i = r! \binom{X}{r}.$$

To define the period Ω_E, let us recall that elliptic curves with complex multiplications have potential good reduction at all non-archimedean places. Hence, up to a finite extension of the field of definition, the space of invariant holomorphic 1-forms on E has an integral structure. Let ω_{int} be a generator of the underlying $\mathcal{O}_K$-module: it is defined up to a unit in $\mathcal{O}_K$. Pick $\tau \in \mathcal{H}$ such that there is an isomorphism of complex tori

$$\phi : \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau \xrightarrow{\sim} E \otimes \mathbf{C}. \tag{3}$$

Then the differential 1-form $\phi^*(\omega_{\mathrm{int}})$ on $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ will be a scalar multiple of the 1-form defined by dz. So, by abuse of notation, we write

$$\phi^*(\omega_{\mathrm{int}}) = \Omega_E \, dz \tag{4}$$

for some $\Omega_E \in \mathbf{C}$. The name of *period* given to Ω_E is justified by the fact that it is an integral period of E, as

$$\Omega_E = \int_0^1 \phi^*(\omega_{\mathrm{int}}) = \int_{\phi([0,1])} \omega_{\mathrm{int}}.$$

Note that choosing a different τ to write the isomorphism (3) results in a different normalization of the period lattice of E: hence the number Ω_E, as defined by (4) will be altered just by a unit in $\mathcal{O}_K$. So the indeterminacy of Ω_E is confined to a unit in $\mathcal{O}_K$.

4. A BRIEF DESCRIPTION OF THE MAAß OPERATORS

Let us recapitulate the essential aspects of the theory of the Maaß differential operators in one variable. For the theory in several variables, which is for many aspects totally analogous, see [2] or [6].

The operators δ_k, $k=0,1,2,\ldots$, on $\mathcal{H}$ were first considered in [8]. Their expression in the coordinate $z=x+iy\in\mathcal{H}$ is

$$\delta_k = -(1/4\pi)\left(2i\frac{d}{dz}+k/y\right). \tag{5}$$

If we extended the definition 1.1 to include C^∞-functions, and denote $G_k^\infty(N)$ the corresponding spaces, then the operator (5) descends to an operator δ_k: $G_k^\infty(N)\to G_{k+2}^\infty(N)$, as an easy computation shows.

In order to obtain algebraicity (and integrality) results, one needs to reinterpret *à la Katz* the operators δ_k in terms of the relative de Rham cohomology of the universal family of elliptic curves with level structure.

Let us then consider the family of elliptic curves $\pi: E_{\mathcal{H}}\to\mathcal{H}$, where $\pi^{-1}(\tau)=$ $=E_\tau=\boldsymbol{C}/\boldsymbol{Z}\oplus\boldsymbol{Z}\tau$. When $N\geq 3$, $\Gamma(N)$ acts without fixed points on $E_{\mathcal{H}}$ defining a smooth family $E_N\to Y(N)$ to which we can attach the relative de Rham cohomology bundle $H^1_{\mathrm{DR}}(E_N/Y(N))$ whose fiber over $\tau\in Y(N)$ is the first de Rham group of E_τ. For simplicity, let us denote H^1_∞ the associated C^∞-bundle. The Hodge decomposition of each fiber induces a splitting $H^1_\infty=H^{1,0}\oplus H^{0,1}$ where $H^{1,0}$ is isomorphic to the line bundle $\underline{\omega}=\pi_*\boldsymbol{\Omega}^1_{E_N/Y(N)}$. Let Split: $H^1_\infty\to\underline{\omega}$ be the resulting projection.

We can now define an operator $\theta_k:\underline{\omega}^{\otimes k}\to\underline{\omega}^{\otimes k+2}$ through the following steps:

Step 1: Embed $\underline{\omega}^{\otimes k}\hookrightarrow\mathrm{Symm}^k(H^1_\infty)$;

Step 2: Use the Gauß-Manin connection $\nabla: H^1_\infty\to H^1_\infty\otimes\boldsymbol{\Omega}^1$ to define a map $\nabla_k:\mathrm{Symm}^k(H^1_\infty)\to\mathrm{Symm}^k(H^1_\infty)\otimes\boldsymbol{\Omega}^1$ by product rule;

Step 3: Apply the Kodaira-Spencer isomorphism $\boldsymbol{\Omega}^1\simeq\underline{\omega}^{\otimes 2}$;

Step 4: Project: $\mathrm{Split}^k:\mathrm{Symm}^k(H^1_\infty)\otimes\underline{\omega}^{\otimes 2}\to\underline{\omega}^{\otimes k}\otimes\underline{\omega}^{\otimes 2}\simeq\underline{\omega}^{\otimes k+2}$.

As explicitly computed in [2], after identifying modular forms of weight k with certain global sections of the line bundle $\underline{\omega}^{\otimes k}$, there is an equality of operators $\delta_k=(-1/4\pi)\,\theta_k$.

One of the advantages of Katz's point of view is that the operators θ_k can be modified, using the very same construction, once an other splitting of the de Rham bundle is available. For instance, working over a p-adic ring, is possible to define the p-adic Maaß operators exploiting the unit root space decomposition. One of the keys to understand the integrality criterion 3.1 is the fact that on the fibers over points of $Y(N)$ corresponding to curves with complex multiplications, the Hodge splitting and the p-adic splitting coincide with the splitting induced by the complex multiplications themselves [5].

For the sake of completeness, it should be mentioned that the Maaß operators are subject to a third interpretation, as elements of a certain universal enveloping algebra [1,2]. As $\mathcal{H}=SL_2(\boldsymbol{R})/K$ where

$$K=\left\{r(\vartheta)=\begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix},\ \vartheta\in\boldsymbol{R}\right\},$$

an element $f \in G_k^\infty$ can be lifted to a C^∞ function $\phi_{k,f}$ on $G = SL_2(\boldsymbol{R})$ by the formula

$$\phi_{k,f}(g) = (cz+d)^{-k} f(g \cdot i), \qquad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Then $\phi_{k,f}$ satisfies the relations

$$(6) \qquad \begin{cases} \phi_{k,f}(\gamma g) = \phi_{k,f}(g), & \forall \gamma \in \Gamma(N) \\ \phi_{k,f}(g\varkappa) = e^{-ik\vartheta} \phi_{k,f}(g), & \forall \varkappa = r(\vartheta) \in K. \end{cases}$$

On the other hand if $\phi \in C^\infty(G)$ satisfies the relations (6) for some k, N, then the formula

$$f_{k,\phi}(z) = (cz+d)^k \phi(g), \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{such that} \quad g \cdot i = z$$

defines an element of $G_k^\infty(N)$.

The Lie algebra of G acts on $C^\infty(G)$ by

$$A * \phi(g) = \frac{d}{dt}\bigg|_{t=0} \phi(g \exp tA), \qquad A \in \mathrm{Lie}(G), \quad \phi \in C^\infty(G).$$

The adjoint action of K induces a decomposition $\mathrm{Lie}(G) \otimes \boldsymbol{C} = \boldsymbol{C}H \oplus \boldsymbol{C}X \oplus \boldsymbol{C}Y$ with

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

and

$$(7) \qquad \mathrm{Ad}(r(\vartheta))X = e^{-2i\vartheta}X, \quad \mathrm{Ad}(r(\vartheta))Y = e^{2i\vartheta}Y.$$

Formulae (7) imply that X defines an operator $D_k : G_k^\infty(N) \to G_{k+2}^\infty(N)$ by $D_k f = f_{k+2, X * \phi_{k,f}}$. After the prescribed identifications, an explicit computation yields $\delta_k = (-1/4\pi) D_k$.

5. The idea of the proof

The integrality criterion (3.1) can be proven showing that the numbers $c_r(f)$ are in fact the coefficients of the power series expansion of f relative to a well-chosen local parameter T at the point $x \in Y(N)$ corresponding to the elliptic curve E under consideration. This parameter T has the following two properties:

(1) T is an eigenvector for the action of the complex multiplications of E on the completion of the local ring at x (*i.e.*, on the fiber at x of the jet bundle on $Y(N)$);

(2) T is rational over the field K, and $\exp(T)$ is v-integral.

The second property is obtained relating T to a formal parameter constructed quite naturally via the classification theory of formal deformations of ordinary elliptic curves in positive characteristic, which is exposed in [7]. Incidentally, the fact that this theory, due to Serre and Tate, holds for ordinary abelian varieties of any dimension, seems to suggest that results similar to Theorem 3.1 should be expected also for Siegel modular forms.

It should also be noted that in order to obtain integrality results, it is essential to consider the p-adic analogous of the Maaß operators mentioned in the previous section.

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