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Marlène Frigon, Antonio Marino, Claudio Saccon

Some problems of parabolic type with discontinuous nonlinearities on convex constraints

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Equazioni a derivate parziali. — *Some problems of parabolic type with discontinuous nonlinearities on convex constraints.* Nota (*) di Marlène Frigon, Antonio Marino e Claudio Saccon, presentata dal Socio E. De Giorgi.

Abstract. — We study semilinear equations and inequalities of parabolic type with discontinuous nonlinearities, possibly subjected to convex or even nonconvex constraint conditions. To prove some existence theorems we regard the solutions as «curves of maximal relaxed slope» for a suitable functional on the given constraint.

Key words: Curves of maximal slope; Partial differential equations; Discontinuity; Nonconvex constraints.

Riassunto. — *Alcuni problemi di tipo parabolico semilineare con termini non lineari discontinui su vincoli non convessi.* Si studiano alcune equazioni e disequazioni di tipo parabolico semilineare con termine non lineare discontinuo, in presenza di condizioni di vincolo anche non convesse. Per ottenere dei teoremi di esistenza si interpretano le soluzioni come «curve di massima pendenza rilassata» per un opportuno funzionale sul vincolo considerato.

Introduction

We deal with a real function $g\colon \mathbb{R} \to \mathbb{R}$, possibly discontinuous, an open subset Ω in $\mathbb{R}^N$ a function $\varphi\colon \Omega \to \mathbb{R}$ (the obstacle) and a real number $\rho > 0$.

We consider the following evolution problems: find absolutely continuous curves $\mathcal{U}\colon I \to L^2(\Omega)$ (I is an interval) such that:

$$\text{(P.1)} \qquad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega) \quad \forall t \text{ in } I \text{ and } a.e. \text{ in } I\colon \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)); \end{cases}$$

$$\text{(P.2)} \qquad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega), \quad \mathcal{U}(t) \geqslant \varphi \qquad a.e. \text{ in } \Omega \ \forall t \text{ in } I \text{ and } a.e. \text{ in } I\colon \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) \qquad a.e. \text{ in } \{x \mid \mathcal{U}(t)(x) > \varphi(x)\}, \\ \mathcal{U}'(t) = [\Delta\mathcal{U}(t) + g(\mathcal{U}(t))]^+ \qquad a.e. \text{ in } \{x \mid \mathcal{U}(t)(x) = \varphi(x)\}, \end{cases}$$

(here the convex constraint $\{u \geqslant \varphi\}$ is involved);

$$\text{(P.3)} \qquad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega), \quad \mathcal{U}(t) \geqslant \varphi \quad a.e. \text{ in } \Omega, \quad \int_\Omega \mathcal{U}(t)^2\,dx = \rho^2 \quad \forall t \text{ in } I \\ \text{and there exists } \Lambda\colon I \to \mathbb{R} \text{ such that } a.e. \text{ in } I\colon \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\,\mathcal{U}(t) \qquad a.e. \text{ in } \{x \mid \mathcal{U}(t)(x) > \varphi(x)\}, \\ \mathcal{U}'(t) = [\Delta\mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\,\mathcal{U}(t)]^+ \qquad a.e. \text{ in } \{x \mid \mathcal{U}(t)(x) = \varphi(x)\}, \end{cases}$$

(here the nonconvex constraints $\{u \geqslant \varphi, \int_\Omega u^2\,dx = \rho^2\}$ is involved).

(*) Pervenuta all'Accademia il 28 agosto 1989.

The main results we get are theorems (1.3), (2.1) and (3.3). We remark that their statements are invariant with respect to the replacement of g with an equivalent function. For the sake of simplicity we assume g not to depend on x in Ω and to have at most a linear growth; such condition could be easily weakened adding some technicality. We also point out that in (P.2), instead of considering the «one-side constraint» $\{u \geqslant \varphi\}$, one could as well take into account the «two obstacle constraint» $\{\varphi_1 \leqslant u \leqslant \varphi_2\}$, and solve the corresponding problem.

We recall that the variational inequalities involved in (P.1) and (P.2) was already studied in [23], with techniques of differential inclusions (see [1], [2]).

We treat this kind of problems with the variational methods introduced in [9], which allowed to face and solve other problems involving non convex constraints (see [5], [6], [10], [11], [17], [19], [21]). Also in this paper we use the concepts of slope and subdifferential (see [9], [12], [18]) and, for each problem (P.1), (P.2) and (P.3), we find solutions as «curves of maximal relaxed slope» for a suitable functional f defined in a Hilbert space H. We point out that:

a) since g is not continuous, f is not φ-convex nor belongs to the classes $\mathcal{H}(H; r, s)$ introduced in [18];

b) we use compactness arguments which would not be required, if f were φ-convex or $f \in \mathcal{H}(H; r, s)$;

c) we obtain existence theorems without uniqueness.

In particular, for what concerns point *a*), we point out that in the problems we are going to consider, we do not have the property:

if $(u_h)_h$, u are such that $u_h \to u$, $f(u_h) \to f(u)$
and $(\alpha_h)_h$, α are such that α_h is in the subdifferential of f at u_h,
$\alpha_h \to \alpha$ weakly, then α is in the subdifferential of f at u;

such a property holds, on the contrary for φ-convex functions and for functions in the classes $\mathcal{H}(H; r, s)$.

1. The unconstrained problem

Let $g: \mathbb{R} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ be as in the introduction.

(1.1) We shall consider the following assumptions on g.

(g.1) there exist a, b in $\mathbb{R}$ such that $|g(s)| \leqslant a + b|s|$, $\qquad \forall s \in \mathbb{R}$;

(g.2) there exists $E \subset \mathbb{R}$ such that $\operatorname{meas}(E) = 0$ and $g|_{\mathbb{R}\setminus E}$ is continuous on $\mathbb{R} \setminus E$.

We wish to remark that the growth condition (g.1) could be further weakened, adding some technical complications. For what concerns (g.2) observe that it is certainly verified if g is a function with bounded variation. We also recall that, for a g bounded on bounded intervals (which is the case, if (g.1) holds), then (g.2) is equivalent to saying that g is Riemann integrable on any bounded intervals.

(1.2) DEFINITION. *We define* $G, \underline{g}, \overline{g}\colon \mathbb{R} \to \mathbb{R}$ as follows

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,,$$

$$\overline{g}(s) = \inf\{h(s) \mid h\colon \mathbb{R} \to \mathbb{R} \textit{ is continuous, } h \geqslant g \textit{ almost everywhere}\}\,,$$

$$\underline{g}(s) = \sup\{h(s) \mid h\colon \mathbb{R} \to \mathbb{R} \textit{ is continuous, } h \leqslant g \textit{ almost everywhere}\}\,.$$

Observe that, if (g.2) *holds, then* $\underline{g}(s) = \overline{g}(s) = g(s) = G'(s)$ *for all* s *in* $\mathbb{R} \setminus E$.

(1.3) THEOREM (Problem (P.1)). *Assume* (g.1) *and* (g.2) *of* (1.1). *Then for all* u_0 *in* $H_0^1(\Omega)$ *there exists* $\mathcal{U}\colon [0, +\infty[\to L^2(\Omega)$, *an absolutely continuous curve such that* $\mathcal{U}(0) = u_0$ *and*

$$a) \quad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega) \quad \forall t \geqslant 0 \textit{ and for a.e. } t \geqslant 0\colon & \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) & \textit{a.e. in } \{x \mid \mathcal{U}(t)(x) \notin E\}\,, \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) = 0 \in [\underline{g}(\mathcal{U}(t)), \overline{g}(\mathcal{U}(t))] & \textit{a.e. in } \{x \mid \mathcal{U}(t)(x) \in E\}\,; \end{cases}$$

b) the function $t \mapsto \frac{1}{2}\int_\Omega |D\mathcal{U}(t)|^2\, dx - \int_\Omega G(\mathcal{U}(t))\, dx$ *is continuous and non increasing;*

c) if, in particular, $\operatorname{essinf} g > 0$ *or* $\operatorname{esssup} g < 0$ *then* $\mathcal{U}$ *is a solution of* (P.1) *and the set* $\{x \mid \mathcal{U}(t)(x) \in E\}$ *is negligible for almost all* $t \geqslant 0$.

2. THE PROBLEM ON A CONVEX CONSTRAINT

Let $g\colon \mathbb{R} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $\varphi\colon \Omega \to \mathbb{R}$ be as in the introduction.

We set $K = \{u \in L^2(\Omega) \mid u \geqslant \varphi \textit{ a.e.} \text{ in } \Omega\}$ and for u in K we define the «contact set» $C(u)$ by $C(u) = \{x \in \Omega \mid u(x) = \varphi(x)\}$.

(2.1) THEOREM (Problem (P.2)). *Assume* (g.1) *and* (g.2) *of* (1.1) *and suppose* $\varphi \in W^{2,2}(\Omega)$.

Then for all u_0 *in* $H_0^1(\Omega) \cap K$ *there exists* $\mathcal{U}\colon [0, +\infty[\to L^2(\Omega)$, *an absolutely continuous curve such that* $\mathcal{U}(0) = u_0$ *and*

$$a) \quad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega) \cap K \quad \forall t \geqslant 0 \textit{ and for a.e. } t \geqslant 0\colon \\ \textit{in } \{x \mid \mathcal{U}(t)(x) \notin E\} \\ \mathcal{U}'(t) = \begin{cases} \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) & \textit{in } \Omega \setminus C(\mathcal{U}(t))\,, \\ [\Delta\mathcal{U}(t) + g(\mathcal{U}(t))]^+ & \textit{in } C(\mathcal{U}(t))\,, \end{cases} \\ \textit{in } \{x \mid \mathcal{U}(t)(x) \in E\} \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) = 0 \in \begin{cases} [\underline{g}(\mathcal{U}(t)), \overline{g}(\mathcal{U}(t))] & \textit{in } \Omega \setminus C(\mathcal{U}(t))\,, \\ [\underline{g}(\mathcal{U}(t)), +\infty[& \textit{in } C(\mathcal{U}(t))\,; \end{cases} \end{cases}$$

if we do not assume that $\varphi \in W^{2,2}(\Omega)$, *then the corresponding variational inequalities hold;*

b) the function $t \mapsto \frac{1}{2}\int_\Omega |D\mathcal{U}(t)|^2\, dx - \int_\Omega G(\mathcal{U}(t))\, dx$ *is continuous and non increasing;*

c) if, in particular, $\operatorname{essinf} g > 0$ *or* $\operatorname{esssup} g < 0$ *then* $\mathcal{U}$ *is a solution of* (P.2). *In the first case* ($\underline{g} > 0$ *in* E) *the set* $\{x \mid \mathcal{U}(t)(x) \in E\}$ *is negligible for almost all* $t \geq 0$.

3. The case with a nonconvex constraint

Let $g\colon \mathbb{R} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $\varphi\colon \Omega \to \mathbb{R}$, $\rho > 0$ be as in the introduction.

We set

$$S_\rho = \left\{ u \in L^2(\Omega) \,\middle|\, \int_\Omega u^2\, dx = \rho^2 \right\}, \quad \rho_K = \left(\int_\Omega (\varphi^+)^2\, dx \right)^{1/2} = \min \left\{ \left(\int_\Omega u^2\, dx \right)^{1/2} \,\middle|\, u \in K \right\}.$$

(3.1) We shall consider the following assumptions.

If $u_0 \in K \cap S_\rho$, we say that (N.T. u_0) holds at u_0, if:

$$(\text{N.T.}\, u_0) \qquad \begin{cases} \displaystyle\int_\Omega u_0^2\, dx > \rho_K^2, \\ \operatorname{meas}(\{x \mid \varphi(x) < u_0(x) < 0\} \cup \{x \mid u_0(x) > 0\}) > 0. \end{cases}$$

Sometimes we require:

$$(\text{N.T.}) \qquad \begin{cases} \rho > \rho_K,\ \varphi \in W^{2,2}(\Omega) \cap C(\Omega), \text{ there esist no } \Omega' \subset \Omega,\ \Omega' \text{ open}, \\ \text{such that } \varphi \leq 0 \text{ in } \Omega',\ \varphi \in H_0^1(\Omega'). \end{cases}$$

The meaning of (N.T. u_0) and (N.T.) is cleared by the following statement (see [6]).

(3.2) Proposition:

a) Let $u_0 \in K \cap S_\rho$. *Then* (N.T. u_0) *holds if and only if* K *and* S_ρ *are not tangent at* u_0, *in the sense that the tangent plane to* S_ρ *at* u_0 *is not tangent to* K.

b) If (N.T.) *holds, then* K *and* S_ρ *are not tangent at any* u *of* $K \cap S_\rho \cap H_0^1(\Omega)$.

(3.3) Theorem (Problem (P.3)). *Assume* (g.1) *and* (g.2) *of* (1.1) *and suppose* $\varphi \in W^{2,2}(\Omega)$.

Then for all u_0 *in* $H_0^1(\Omega) \cap K$ *such that* (N.T. u_0) *holds at* u_0 *there exist* $T > 0$, $\mathcal{U}\colon [0, T[\to L^2(\Omega)$, *an absolutely continuous curve and* $\Lambda\colon [0, T[\to \mathbb{R}$ *such that* $\mathcal{U}(0) = u_0$ *and*

$$a) \quad \begin{cases} \mathcal{U}(t) \in H_0^1(\Omega) \cap K \cap S_\rho \quad \forall t \in [0, T[\textit{ and for a.e. in } [0, T[: \\ \textit{in } \{x \mid \mathcal{U}(t)(x) \notin E\} \\ \mathcal{U}'(t) = \begin{cases} \Delta \mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\, \mathcal{U}(t) & \textit{in } \Omega \setminus C(\mathcal{U}(t)), \\ [\Delta \mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\, \mathcal{U}(t)]^+ & \textit{in } C(\mathcal{U}(t)), \end{cases} \\ \textit{in } \{x \mid \mathcal{U}(t)(x) \in E\} \\ \mathcal{U}'(t) = \Delta \mathcal{U}(t) = 0 \in \begin{cases} [\underline{g}(\mathcal{U}(t)) + \Lambda(t)\, \mathcal{U}(t),\ \overline{g}(\mathcal{U}(t)) + \Lambda(t)\, \mathcal{U}(t)] & \textit{in } \Omega \setminus C(\mathcal{U}(t)), \\ [\underline{g}(\mathcal{U}(t)) + \Lambda(t)\, \mathcal{U}(t),\ +\infty[& \textit{in } C(\mathcal{U}(t)); \end{cases} \end{cases}$$

if we do not assume that $\varphi \in W^{2,2}(\Omega)$, *then the corresponding variational inequalities hold; if* (N.T.) *is assumed, then* $T = +\infty$;

b) the function $t \mapsto \frac{1}{2}\int_\Omega |DU(t)|^2\,dx - \int_\Omega G(\mathcal{U}(t))\,dx$ *is continuous and non increasing;*

c) if, in particular, we have

$$\varphi \leqslant 0\,, \qquad \frac{g(s)}{s} \leqslant \Sigma \qquad \textit{with} \qquad \Sigma < \inf\left\{ \frac{\int_\Omega |Du|^2\,dx}{\int_\Omega u^2\,dx} \;\middle|\; u \in H_0^1(\Omega) \right\},$$

$$g(0) = 0\,, \qquad g(s)\,s \geqslant 0 \qquad (\textit{it would suffice } \bar{g}(s)\,s \geqslant 0 \textit{ and } gs(s) \geqslant 0 \textit{ in } E)\,,$$

then it can be proved that $\mathcal{U}$ *solves problem* (P.3).

4. The curves of maximal relaxed slope

The abstract framework, in which the theorems of the previous sections can be proved, can be divided into two parts. In the first one, which is treated in this section, we introduce the notion of curve of maximal relaxed slope and give an existence theorem under quite general assumptions. In the second one, which is in section 5, we show that, under suitable conditions, such curves solve an evolution equation similar to the classical $\mathcal{U}' = -\operatorname{grad} f \circ \mathcal{U}$.

Let (X, d) be a metric space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$, be a function. We set $\mathcal{D}(f) = \{u \in X \,|\, f(u) < +\infty\}$. Sometimes we shall consider in $\mathcal{D}(f)$ the «graph metric» d^* defined by $d^*(v, u) = d(v, u) + |f(v) - f(u)|$.

(4.1) Definition. *Let* $u \in \mathcal{D}(f)$. *We set* $\chi_u(\rho) = \inf\{f(v) \,|\, d(v, u) \leqslant \rho\}$, *for* $\rho \geqslant 0$. *We define (see* [9]) *the «slope of f at u», denoted by* $|\nabla f|(u)$, *by*

$$|\nabla f|(u) = -\liminf_{\rho \to 0^+} [\chi_u(\rho) - \chi_u(0)]\,\rho^{-1}\,.$$

We define the «relaxed slope of f at u», denoted by $\overline{|\nabla f|}(u)$ *by*

$$\overline{|\nabla f|}(u) = \liminf_{d^*(v,u) \to 0} |\nabla f|(v)\,.$$

(4.2) Definition. *Let I be an interval with non empty interior and let* $\mathcal{U}\colon I \to X$ *be a curve. We say that* $\mathcal{U}$ *is a curve of maximal relaxed slope almost everywhere for f (see* [9] *and* [18]) *if*

a) $\mathcal{U}$ *is continuous;*

b) $f \circ \mathcal{U}(t) < \infty \quad \forall t$ *in I with* $I > \inf I$,
$f \circ \mathcal{U}(t) \leqslant f \circ \mathcal{U}(t_0) \quad \forall t$ *in I if there exists* $t_0 = \min I$;

c) $d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leqslant \int_{t_1}^{t_2} \overline{|\nabla f|}(\mathcal{U}(t))\,dt \quad \forall t_1, t_2$ *in I with* $t_1 \leqslant t_2$;

d) there exists $\tilde{f}\colon I \to \mathbb{R} \cup \{+\infty\}$, *equivalent to* $f \circ \mathcal{U}$, *such that*

$$\tilde{f}(t_2) - \tilde{f}(t_1) \leqslant -\int_{t_1}^{t_2} \left(\overline{|\nabla f|}(\mathcal{U}(t))\right)^2 dt \qquad \forall t_1, t_2 \textit{ in } I \textit{ with } t_1 \leqslant t_2\,.$$

If d) holds with $\tilde{f} = f \circ \mathcal{U}$, *we say that* $\mathcal{U}$ *is a curve of maximal relaxed slope for f.*

(4.3) Remark. *It is easy to see that*

1) $|\overline{\nabla f}|(\mathcal{U}(t)) < +\infty$ *for a.e. t in I;*

2) *$\mathcal{U}$ is absolutely continuous on any compact subsets of* $I \setminus \inf I$ *(of I if I has a minimum and* $f(\min I) < \infty$).

(4.4) Remark. *If, for instance, X is an open subset of a Hilbert space H and f is a C^1 function, then $\mathcal{U}$ is a curve of maximal relaxed slope almost everywhere for f if and only if* $\mathcal{U}'(t) = -\operatorname{grad} f(\mathcal{U}(t))$ $\forall t$ *in I.*

If f is a convex function, then $\mathcal{U}$ is a curve of maximal relaxed slope almost everywhere for f if and only if (*see* [3]) $\mathcal{U}'(t) \in \partial f(\mathcal{U}(t))$ *a.e. in I.*

For the existence and for a first regularity theorem we need the following definitions.

(4.5) Definition. *We say that f is «∇-continuous» if for all u in $\mathcal{D}(f)$ and for all C in $\boldsymbol{R}$, we have*

$$\lim_{\substack{v \to u \\ f(v) \leq C,\, |\overline{\nabla f}|(v) \leq C}} f(v) = f(u)\,.$$

We say that f is «$d\overline{\nabla}$-continuous», if for all u in $\mathcal{D}(f)$, for all sequences $(u_n)_n$ in $\mathcal{D}(f)$ converging to u and such that

$$\sup_n f(u_n) < +\infty\,, \qquad \lim_n d(u_n, u)\,|\overline{\nabla f}|(u_n) = 0\,,$$

we have $\lim_n f(u_n) = f(u)$.

It is easy to see that $f\, d\overline{\nabla}$-continuous $\Rightarrow f$ ∇-continuous.

We also need the following compactness assumption.

(4.6) Definition. *We say that f is coercive in X if for all C in $\mathbb{R}$ the set $\{v \in X \mid f(v) \leq C\}$ is compact. We say that f is locally coercive, if for all u in X there exists $\rho > 0$ such that f is coercive in $\{v \in X \mid d(v, u) \leq \rho\}$.*

(4.7) Theorem (existence). *Assume that*

a) f is locally coercive;

b) f is ∇-continuous.

Then for all u_0 in $\mathcal{D}(f)$ there exist $T > 0$ and $\mathcal{U}\colon [0, T[\to X$ a curve of maximal relaxed slope almost everywhere for f such that $\mathcal{U}(0) = u_0$.

The proof of (4.7) is essentially given in theorem (4.10) of [18] (see also [15]).

The following proposition adds some informations on the behaviour of f along the curve $\mathcal{U}$ (see [15]).

(4.8) Proposition. *Assume that f is $d\overline{\nabla}$-continuous and let $\mathcal{U}\colon I \to X$ be a curve of maximal relaxed slope almost everywhere for f. Then $f \circ \mathcal{U}$ is continuous, hence it is non increasing and $\mathcal{U}$ is a curve of maximal relaxed slope for f.*

We introduce now a class of functions which are $d\overline{\nabla}$-continuous. Let H be a Hilbert space and M be a smooth manifold in H.

(4.9) DEFINITION. *If K is a convex subset of H, we say that M and K are not tangent, if for all u in $K \cap M$ the tangent plane to M at u is not tangent to K.*

We shall denote by I_M the function with value 0 on M and value $+\infty$ outside.

(4.10) THEOREM. *Let X be an open subset of H, $f_0\colon H \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function and $h\colon M \to \mathbb{R}$ a locally lipschitzian function. Then*

a) $f_0 + h$ is $d\overline{\nabla}$-continuous;

b) if M is C^1 and has finite codimension, if $\mathcal{D}(f_0)$ and M are non tangent, then $f_0 + h + I_M$ is $d\overline{\nabla}$ continuous and

$$|\nabla(f_0 + h)|(u) \leqslant a(u) + b(u)|\nabla(f_0 + h + I_M)|(u) \qquad \forall u \text{ in } \mathcal{D}(f_0) \cap M,$$

where a and b are suitable continuous functions on $\mathcal{D}(f_0) \cap M$.

We remark briefly that *a*) and *b*) follow from the fact that, if f is as above, then the inequality

$$f(v) \geqslant f(u) - \Psi(u, v)(1 + |\nabla f|(u))\|v - u\| \qquad \forall u, v \text{ in } \mathcal{D}(f) \text{ with } |\nabla f|(u) < +\infty$$

holds (for a suitable continuous function Ψ) and therefore f is $d\overline{\nabla}$-continuous.

5. AN EVOLUTION EQUATION

To study problems concerning parabolic equations and inequalities, like those considered in the first three sections, it is important to find conditions ensuring that the curve of maximal relaxed slope satisfy an equation analogous to the classical $\mathcal{U}' = -\operatorname{grad} f \circ \mathcal{U}$.

To this aim we shall introduce some operators that play the role of the gradient of f for non regular f's.

Let H be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, let X be a subset of H and $f\colon X \to \boldsymbol{R} \cup \{+\infty\}$ be a given function. Remember that $\mathcal{D}(f) = \{u \in X \mid f(u) < +\infty\}$.

(5.1) DEFINITION. *Let $\mathcal{A}$ be a multivalued map, $\mathcal{A}\colon \mathcal{D}(f) \to 2^H$. We say that $\mathcal{A}$ is a «subdifferential along curves for f», if*

$$\begin{cases} \text{for all absolutely continuous curves } \mathcal{U}\colon I \to X \text{ such that} \\[1ex] \qquad \sup\limits_{t \in I} f \circ \mathcal{U}(t) < +\infty, \quad \mathcal{A}(\mathcal{U}(t)) \neq \emptyset \quad \text{a.e. in } I, \\[1ex] \text{one has for almost all } t \text{ in } I \\[1ex] \qquad \liminf\limits_{h \to 0^+} [f \circ \mathcal{U}(t+h) - f \circ \mathcal{U}(t)]\, h^{-1} \geqslant \langle \alpha, \mathcal{U}'(t) \rangle \quad \forall \alpha \text{ in } \mathcal{A}(\mathcal{U}(t)). \end{cases}$$

The following lemma establishes the link between the curves of maximal relaxed slope and an equation of the type written above.

(5.2) LEMMA. *Let $\mathcal{D}(\overline{|\nabla f|}) = \{u \in \mathcal{D}(f) \mid \overline{|\nabla f|}(u) < +\infty\}$ and suppose that $A\colon \mathcal{D}(\overline{|\nabla f|}) \to H$ is an operator such that* *a) A is a subdifferential along curves for f;*

b) $\|A(u)\| \leq \overline{|\nabla f|}(u) \quad \forall u$ in $\mathcal{D}(\overline{|\nabla f|})$.

Then, if $\mathcal{U}\colon I \to X$ is a curve of maximal relaxed slope almost everywhere for f, one has for a.e. t in I

$$(E)\qquad \begin{cases} \mathcal{U}'(t) = -A(\mathcal{U}(t)), \\ \tilde{f}'(t) = -\|\mathcal{U}'(t)\|^2, \\ \|A(\mathcal{U}(t))\| = \overline{|\nabla f|}(\mathcal{U}(t)), \end{cases}$$

($\tilde{f}$ is a non increasing function equivalent to $f \circ \mathcal{U}$, which does exist by d) of definition (4.2)).

Now we show how one can find an operator A satisfying *b)* of lemma (5.2). We recall the definition of subdifferential.

(5.3) Definition. *Let $u \in \mathcal{D}(f)$, we define the «subdifferential» of f at u as the set $\partial^- f(u)$ of all α in H such that*

$$\liminf_{v \to u} [f(v) - f(u) - \langle \alpha, v - u \rangle] \|v - u\|^{-1} \geq 0 .$$

It can be easily seen that $\partial^- f(u)$ is closed and convex; if $\partial^- f(u) \neq \emptyset$, we can define the subgradient of f at u, denoted by $\operatorname{grad}^- f(u)$, as the unique element of $\partial^- f(u)$ which has minimal norm.

We remark that, as one can easily see, $|\nabla f|(u) \leq \|\alpha\|$ for all α in $\partial^- f(u)$. In general it can happen that $|\nabla f|(u) < \|\operatorname{grad}^- f(u)\|$, but if f is smooth or it is convex (see [18]), then the equality holds. It is also evident that, for two given functions $f, g\colon H \to \mathbb{R} \cup \{+\infty\}$, we have $\partial^- f(u) + \partial^- g(u) \subset \partial^-(f+g)(u)$ and that the equality holds, if g is differentiable.

A result, which is useful to understand the properties of the subdifferential and the forecoming definitions, is the following one.

(5.4) Proposition. *If f is lower semicontinuous, then the set $\{u \in \mathcal{D}(f) \mid \partial^- f(u) \neq \emptyset\}$ is dense in $\mathcal{D}(f)$.*

If f is locally coercive (which is the main assumption of this paper), the above result is easily proved by considering, for u in $\mathcal{D}(f)$, the minimizers of the function $v \mapsto f(v) + k\|v - u\|^2$, for k large.

We also remark that the subdifferential certainly is a subdifferential along curves for f (see lemma (1.10) of [18]), but in general does not satisfy *b)* of lemma (5.2) (it would if f were φ-convex): this is the situation arising in the study of the problems of sections 1, 2 and 3. The operators that we are going to introduce are built up so as to verify *b)* of (5.2); in the concrete problems they also are subdifferentials along curves.

(5.5) Definition. *We introduce the multivalued map, $\mathcal{A}(f)\colon \mathcal{D}(f) \to 2^H$, defined* *by*:

$$\alpha \in \mathcal{A}(f)(u) \Leftrightarrow \begin{cases} \text{there exist a sequence } (u_n)_n \text{ in } \mathcal{D}(f), \text{ such that} \\ \qquad \lim_{n\to\infty} u_n = u, \quad \lim_{n\to\infty} f(u_n) = f(u) \\ \text{and a sequence } (\alpha_n)_n \text{ in } H \text{ such that} \\ \qquad \alpha_n \in \partial^- f(u_n) \quad \forall n, \qquad \alpha_n \to \alpha \text{ weakly in } H. \end{cases}$$

(5.6) LEMMA. *If f is locally coercive (see definition* (4.6)), *then for all u in $\mathcal{D}(f)$ with $\overline{|\nabla f|}(u) < +\infty$ one has*:

a) there exists α in $\mathcal{A}(f)(u)$ such that $\|\alpha\| \leq \overline{|\nabla f|}(u)$;

b) $\overline{|\nabla f|}(u) = \liminf\limits_{\substack{d^*(v,u)\to 0 \\ \partial^- f(v)\neq\emptyset}} \|\mathrm{grad}^- f(v)\|$.

As a consequence we have:

c) f is ∇-continuous (see definition (4.5)), *if and only if, for all u in $\mathcal{D}(f)$*:

$$\liminf_{\substack{v\to u \\ f(v)\leq C,\ \partial^- f(v)\neq\emptyset,\ \|\mathrm{grad}^- f(v)\|\leq C}} f(v) = f(u), \qquad \forall C \text{ in } \mathbf{R};$$

d) f is $d\overline{\nabla}$-continuous (see definition (4.5)), *if and only if, for all u in $\mathcal{D}(f)$, for all sequences $(u_n)_n$ in $\mathcal{D}(f)$ such that*:

$$u_n \to u, \quad \sup_n f(u_n) < +\infty, \quad \partial^- f(u_n) \neq \emptyset \ \forall n, \quad \lim_{n\to\infty} \|\mathrm{grad}^- f(u_n)\| \|u_n - u\| = 0,$$

one has: $\lim_{n\to\infty} f(u_n) = f(u)$.

The results of sections 4 and 5 give the following theorem.

(5.7) THEOREM. *Assume f to be locally coercive, ∇-continuous (see* (5.6) *c)) and $\mathcal{A}(f)$ to be a subdifferential along curves for f.*

Then, for all u_0 in $\mathcal{D}(f)$, there exist $T > 0$ and an absolutely continuous curve $\mathcal{U}: [0, T[\to H$, such that $\mathcal{U}$ is a curve of maximal relaxed slope for f, $\mathcal{U}(0) = u_0$ and:

a) $f \circ \mathcal{U}$ is equivalent to a non increasing function $\tilde{f}: [0, T[\to \mathbf{R} \cup \{+\infty\}$ such that: $\tilde{f}'(t) = -\|\mathcal{U}'(t)\|^2$ for a.e. t in $[0, T[$; if f is $d\overline{\nabla}$-continuous (see (5.6) *d)), then $f \circ \mathcal{U} = \tilde{f}$ is continuous*;

b) for a.e. t in $[0, T[$ $\mathcal{A}(f)(\mathcal{U}(t)) \neq \emptyset$, $\mathcal{A}(f)$ has in $\mathcal{U}(t)$ a unique minimal section, denoted by $A(\mathcal{U}(t))$, and $\mathcal{U}'(t) = A(\mathcal{U}(t))$;

c) moreover, if $\mathcal{A}: \mathcal{D}(f) \to 2^H$ is a multivalued map with the properties:

$$\begin{cases} \mathcal{A}(f)(u) \subset \mathcal{A}(u) \quad \forall u \text{ in } \mathcal{D}(f), \\ \mathcal{A} \text{ is a subdifferential along curves for } f, \end{cases} \tag{5.8}$$

then, for a.e. t $\mathcal{A}(\mathcal{U}(t)) \neq \emptyset$ and $A(\mathcal{U}(t))$ is its unique minimal section.

For what concerns constrained problems, let us consider a smooth manifold M, contained in H; we denote by $N: M \to 2^H$ the multivalued map defined by: $N(u) = \{v \in H \mid v \text{ is orthogonal to } M \text{ at } u\}$.

(5.9) LEMMA. *Let $f = f_0 + h$, where $f_0: H \to \mathbb{R} \cup \{+\infty\}$ is a convex function and h is a locally lipschitzian real function, defined in an open subset X of H.*

Assume f to be locally coercive, M to be a C^1 manifold with finite codimension, M and $\mathcal{D}(f)$ not to be tangent (*see* (4.9)).

Then:

a) $\mathcal{A}(f + I_M)(u) \subset \mathcal{A}(f)(u) + N(u)$ *for every u in* $\mathcal{D}(f) \cap M$;

b) *as a consequence, if $\mathcal{A}: \mathcal{D}(f) \to 2^H$ is a multivalued map such that* (5.8) *hold for the pair $\mathcal{A}$, f, then* (5.8) *hold for the pair $\mathcal{A} + N$, $f + I_M$.*

The following theorem is an immediate consequence of the previous lemma.

(5.10) THEOREM. *Suppose that f and M satisfy the assumptions of lemma* (5.9) *and that $\mathcal{A}(f)$ is a subdifferential along curves for f. Then, for every u_0 in $\mathcal{D}(f + I_M) = \mathcal{D}(f) \cap M$, there exist $T > 0$ and an absolutely continuous curve $\mathcal{U}: [0, T[\to H$ such that $\mathcal{U}$ is a curve of maximal relaxed slope for f* (see (4.2)) *with $\mathcal{U}(0) = u_0$.*

Moreover:

a) $f \circ \mathcal{U}$ *is continuous and* $(f \circ \mathcal{U})'(t) = -\|\mathcal{U}'(t)\|^2$ *a.e. in* $[0, T[$;

b) *for a.e. t in $[0, T[$ $\mathcal{A}(f + I_M)(\mathcal{U}(t)) \neq \emptyset$ and $-\mathcal{U}'(t)$ is its unique minimal section; in particular there exist $\nu(\mathcal{U}(t))$ in $N(\mathcal{U}(t))$ and $\tilde{\mathcal{A}}(\mathcal{U}(t))$ in $\mathcal{A}(f)(\mathcal{U}(t))$ with the property* $\mathcal{U}'(t) = -\tilde{\mathcal{A}}(\mathcal{U}(t)) + \nu(\mathcal{U}(t))$;

c) *furthermore, if $\mathcal{A}: \mathcal{D}(f) \to 2^H$ is a multivalued map satisfying* (5.8), *then for a.e. t in $[0, T[$ $\mathcal{A}(\mathcal{U}(t)) \neq \emptyset$, and $\tilde{\mathcal{A}}(\mathcal{U}(t)) + \nu(\mathcal{U}(t))$ is the unique minimal section of $\mathcal{A}(\mathcal{U}(t)) + N(\mathcal{U}(t))$.*

6. SKETCH OF THE PROOFS

We wish to show the functionals and the operators involved in problems (P.1), (P.2) and (P.3). As in sections 1, 2 and 3 let Ω be a bounded open subset of $\mathbb{R}^N$, $g: \mathbb{R} \to \mathbb{R}$, $\varphi: \Omega \to \mathbb{R}$ given functions and $\rho > 0$ a real number.

We consider the Hilbert space $H = L^2(\Omega)$ with the usual inner product $\langle u, v \rangle = \int_\Omega uv\,dx$ and norm $\|u\|^2 = \int_\Omega u^2\,dx$.

Assuming (g.1) of (1.1) we can introduce the functionals f_1, f_2, $f_3: L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$f_1(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Du|^2\,dx - \int_\Omega G(u)\,dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega); \end{cases}$$

$$f_2 = f_1 + I_K, \qquad f_3 = f_2 + I_{S_\rho}$$

where

$$K = \{u \in L^2(\Omega) \mid u \geq \varphi \text{ a.e. in } \Omega\}, \qquad S_\rho = \left\{u \in L^2(\Omega) \,\middle|\, \int_\Omega u^2\,dx = \rho^2\right\}$$

(in general if $V \subset H$, then I_V is the function which has value 0 on V and $+\infty$ outside).

Moreover, if (g.1) of (1.1) holds, we can define the multivalue maps $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows:

$$\begin{cases} \mathcal{A}_1: H_0^1(\Omega) \to 2^{L^2(\Omega)}, \\ \alpha \in \mathcal{A}_1(u) \Leftrightarrow \int_\Omega Du\, D(v-u)\, dx - \int_\Omega [\underline{g}(u)(v-u)^+ - \overline{g}(u)(v-u)^-]\, dx \geqslant \\ \qquad \geqslant \int_\Omega \alpha(v-u)\, dx \qquad \forall v \text{ in } H_0^1(\Omega); \end{cases}$$

$$\begin{cases} \mathcal{A}_2: H_0^1(\Omega) \cap K \to 2^{L^2(\Omega)}, \\ \alpha \in \mathcal{A}_2(u) \Leftrightarrow \int_\Omega Du\, D(v-u)\, dx - \int_\Omega [\underline{g}(u)(v-u)^+ - \overline{g}(u)(v-u)^-]\, dx \geqslant \\ \qquad \geqslant \int_\Omega \alpha(v-u)\, dx \qquad \forall v \text{ in } H_0^1(\Omega) \cap K. \end{cases}$$

(6.1) *Sketch of the proofs of theorems* (1.3), (2.1) *and* (3.3). Theorems (1.3) and (2.1) can be easily obtained from theorem (5.7) and *a*) of (4.10), just considering the functionals f_1 and f_2 and the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. The solutions are curves of maximal relaxed slope for f_1 and f_2 respectively.

Theorem (3.3) is obtained from theorem (5.10), considering the functional f_2 the manifold S_ρ and the operator $\mathcal{A}_2$. The solutions are curves of maximal relaxed slope for f_3.

7. Some open problems

(7.1) Can (g.2) be removed? We remark that the existence theorem (4.7) gives the existence of a curve of maximal relaxed slope for the functionals f_1, f_2 and f_3 (defined in section 6) even if the assumption (g.2) of (1.1) (g continuous almost everywhere) does not hold. Assumption (g.2) is needed only for ensuring that such curves are solutions of the corresponding equations (P.1), (P.2) and (P.3). One can ask oneself whether this fact is true without assuming (g.2). We remark that some results concerning differential inclusions, which are in some sense generalizations of (P.1) and (P.2), were proved in [23] with no use of (g.2).

It is not clear, however, whether the solutions found in [23] are the same that one finds using theorem (4.7).

(7.2) Uniqueness. It is easy to see that, in general, solutions of (P.1) with a given initial datum u_0 are not unique: take for instance $\Omega =]0, 1[$, $g(s) = \operatorname{sgn}(s)$ and $u_0 = 0$, then there are infinitely many solutions.

One could ask oneself whether it is possible to establish conditions on the solutions or individuate a class of initial data such that the solutions are unique.

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M. Frigon: University of Montreal - Montreal (Canada)
A. Marino e C. Saccon: Dipartimento di Matematica
Università degli Studi di Pisa - Via Buonarroti, 2 - 56100 Pisa