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## Holomorphic automorphism groups in certain compact operator spaces

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**Geometria.** — *Holomorphic automorphism groups in certain compact operator spaces.* Nota di Carlo Petronio, presentata (*) dal Socio E. Vesentini.

Abstract. — A class of Banach spaces of compact operators in Hilbert spaces is introduced, and the holomorphic automorphism groups of the unit balls of these spaces are investigated.

Key words: Compact operator; Unit ball; Isometry; Non-homogeneity.

Riassunto. — *Gruppi di automorfismi olomorfi in certi spazi di operatori compatti.* Viene introdotta una classe di spazi di Banach di operatori compatti tra spazi di Hilbert e viene indagato il gruppo degli automorfismi olomorfi delle palle unitarie corrispondenti.

Let $H$ and $K$ be complex Hilbert spaces. We denote by $\mathcal{L}(H,K)$ the complex Banach space of all bounded linear operators from $H$ to $K$ and by $\mathcal{L}_0(H,K)$ the subspace of $\mathcal{L}(H,K)$ consisting of compact operators. We write $\mathcal{L}(H)$ and $\mathcal{L}_0(H)$ instead of $\mathcal{L}(H,H)$ and $\mathcal{L}_0(H,H)$.

A theory of normed ideals in $\mathcal{L}(H)$ was first sistematically introduced by Schatten in [8] and [9], leading to the definition of a class of subspaces $\mathcal{L}_p(H)$ of $\mathcal{L}_0(H)$ (for $1 \leqslant p < \infty$), which are Banach spaces with respect to a suitable norm. In the first section of this paper this definition will be slightly generalized, introducing a class of subspaces of $\mathcal{L}_0(H,K)$, denoted by $\mathcal{L}_p(H,K)$ (for $1 \leqslant p < \infty$). In sections, 2, 3 and 4 we examine the holomorphic automorphism group of the unit ball of these spaces. Our main result can be considered as an operator analogue of the theorem proved by Vesentini in [11] and [12] about the total non-homogeneity of the unit ball of an $L^p$ space (provided $p \neq 2$, $\infty$ and the space is not isomorphic to $\mathbf{C}$).

**1.** The inner product and the norm will be denoted respectively by $(\cdot\,|\,\cdot)$ and $\|\cdot\|$ in both $H$ and $K$; the inner product will be linear in the first argument and anti-linear in the second. If $\phi \in H$ and $\psi \in K$ we define an operator $\psi \otimes \overline{\phi} \in \mathcal{F}(H,K)$ (the space of finite-rank operators) by $(\psi \otimes \overline{\phi})(\phi_1) = (\phi_1|\phi) \cdot \psi$.

As is well-known (see *e.g.* [3, pp. 68-69]), every $T \in \mathcal{L}(H,K)$ has a unique polar decomposition $T = U[T]$, where $U \in \mathcal{L}(H,K)$ is a partial isometry, $\operatorname{Ker}(U) = \operatorname{Ker}(T)$ and $[T] \in \mathcal{L}(H)$ is the unique positive square root of the positive hermitian operator $T^*T$.

Suppose now $T$ is compact; since $[T] = U^*T$, $[T]$ is compact too, and therefore it is diagonalizable (see *e.g.* [3, pp. 86-87]): if we denote by $\{\mu_n(T)\}$ the sequence of all non-zero eigenvalues of $[T]$ repeated according to their geometric multiplicity and arranged in a non-increasing way, there exists an orthonormal sequence $\{\phi_n\} \subset H$ such that

$$[T] = \sum_n \mu_n(T)\, \phi_n \otimes \overline{\phi}_n$$

(*) Nella seduta del 18 novembre 1989.

for the norm-convergence (of course sequences are allowed to be finite). Since all the $\phi_n$'s belong to the initial space of $U$, the sequence $\{U\phi_n\}$ is orthonormal too. It follows that $T$ can be written in the so-called «canonical form»:

$$T = \sum_n \mu_n(T)\, \psi_n \otimes \overline{\phi}_n$$

where $\{\psi_n\} \subset K$ is an orthonormal sequence.

Now, for $T \in \mathcal{L}(H,K)$ and $1 \leqslant p < \infty$ we define $\|T\|_p \in [0,\infty]$ by

$$\|T\|_p = \begin{cases} \left(\sum \mu_n(T)^p\right)^{1/p} & \text{if } T \in \mathcal{L}_0(H,K), \\ \infty & \text{otherwise} \end{cases}$$

and we define $\mathcal{L}_p(H,K) = \{T \in \mathcal{L}(H,K) : \|T\|_p < \infty\}$. It is easily verified that $\mathcal{L}_p(H,K) \supseteq \mathcal{F}(H,K)$.

The proof of the following theorem imitates closely the argument given *e.g.* in [6] when $K = H$ (cf. [7] for details):

THEOREM 1. For $1 \leqslant p < \infty$ the natural linear structure and the map $\|\cdot\|_p$ define on $\mathcal{L}_p(H,K)$ a complex Banach space structure (a complex Hilbert space structure for $p = 2$).

The introduction of these spaces is strongly related with the problem of defining a trace in the infinite-dimensional case. We briefly mention an equivalent definition of $\|\cdot\|_p$ which exploits this concept (for all the proofs we refer to [7]).

PROPOSITION 1. If $T \in \mathcal{L}(H)$, $T \geqslant 0$ and $\{\phi_\alpha\}_{\alpha\in A}$ is an orthonormal basis of $H$, the (finite or infinite) sum of the positive-term series

$$\sum_{\alpha\in A} (T\phi_\alpha | \phi_\alpha)$$

is independent of the choice of the basis. This sum will be indicated by $\mathrm{tr}(T)$ and called the trace of $T$.

Now let $T \in \mathcal{L}(H,K)$; since $[T] \in \mathcal{L}(H)$ is a positive operator, it can be raised to any positive real power, and the outcome is a positive operator again. We can then state the following:

PROPOSITION 2. $\|T\|_p = (\mathrm{tr}([T]^p))^{1/p}$.

The proof of completeness in Theorem 1 is achieved in [7] by means of another result which has an independent interest. We begin with:

PROPOSITION 3. If $T \in \mathcal{L}_1(H)$ and $\{\phi_\alpha\}_{\alpha\in A}$ is an orthonormal basis of $H$, the series

$$\sum_{\alpha\in A} (T\phi_\alpha | \phi_\alpha)$$

is absolutely convergent and its sum is independent of the choice of the basis. Once again this sum will be called the trace of $T$.

According to this proposition, $\mathcal{L}_1(H)$ is often called the «trace class on $H$».

THEOREM 2. If one of the following hypothesis holds:

*a*) $T \in \mathcal{L}_0(H,K)$, $S \in \mathcal{L}_1(K,H)$, *b*) $T \in \mathcal{L}_1(H,K)$, $S \in \mathcal{L}(K,H)$, *c)* $T \in \mathcal{L}_p(H,K)$, $S \in \mathcal{L}_q(K,H)$ $(1<p, q<\infty,\ 1/q+1/p=1)$

then $TS \in \mathcal{L}_1(K)$, $ST \in \mathcal{L}_1(H)$ and $\operatorname{tr}(TS) = \operatorname{tr}(ST)$.

The following isometric isomorphisms hold:

*a)* $\mathcal{L}_0(H,K)^* \cong \mathcal{L}_1(K,H)$, *b)* $\mathcal{L}_1(H,K)^* \cong \mathcal{L}(K,H)$, *c)* $\mathcal{L}_p(H,K)^* \cong \mathcal{L}_q(K,H)$

the action of an operator $S$ being defined in any case by $S: T \mapsto \operatorname{tr}(TS)$.

**2.** We investigate now the group of all holomorphic automorphisms of the open unit ball of the spaces $\mathcal{L}_p(H,K)$ for $p=0$ and $1 \leqslant p < \infty$.

We begin with $\mathcal{L}_0(H,K)$; since it is a norm-closed ideal in $\mathcal{L}(H,K)$, it is in particular a $J^*$-algebra (see [4]) and therefore its unit ball is homogeneous: as Harris showed in [4] the group of all Möbius transformations operates transitively on it. Hence we only have to determine the isotropy group of the origin, *i.e.* the group of all linear isometries of $\mathcal{L}_0(H,K)$ onto itself.

We will denote the group of all linear isometries of a normed space $F$ onto itself by $\mathfrak{J}(F)$. In [1] Franzoni proved the following:

THEOREM. 1) If $\dim_{\mathbf{C}} H \neq \dim_{\mathbf{C}} K$ then

$$\mathfrak{J}(\mathcal{L}(H,K)) = \{T \mapsto UTV: U \in \mathfrak{J}(K),\ V \in \mathfrak{J}(H)\}.$$

2) If $\tau$ is a fixed transposition on $\mathcal{L}(H)$ then

$$\mathfrak{J}(\mathcal{L}(H)) = \{T \mapsto UTV: U, V \in \mathfrak{J}(H)\} \cup \{T \mapsto U\tau(T)\,V: U, V \in \mathfrak{J}(H)\}.$$

Since two Hilbert spaces having the same complex dimension can be regarded as identical, the above theorem determines $\mathfrak{J}(\mathcal{L}(H,K))$ in every case. We can now prove the following:

THEOREM 3. Every element of $\mathfrak{J}(\mathcal{L}_0(H,K))$ is the restriction of an element of $\mathfrak{J}(\mathcal{L}(H,K))$, and conversely.

PROOF. Since it can be verified directly from Franzoni's theorem that $T \in \mathcal{L}_0(H,K)$, $j \in \mathfrak{J}(\mathcal{L}(H,K)) \Rightarrow j(T) \in \mathcal{L}_0(H,K)$, we only have to prove that every element of $\mathfrak{J}(\mathcal{L}_0(H,K))$ can be extended to an element of $\mathfrak{J}(\mathcal{L}(H,K))$. It follows from theorem 2 that the natural inclusion $\mathcal{L}_0(H,K) \subset \mathcal{L}(H,K)$ is the inclusion of a Banach space into its bi-dual space; given $j \in \mathfrak{J}(\mathcal{L}_0(H,K))$ we have $j^{**} \in \mathfrak{J}(\mathcal{L}(H,K))$, $j = j^{**}|_{\mathcal{L}_0(H,K)}$ and the proof is complete. $\square$

**3.** In this section we prove the main result of the present paper, *i.e.* the total non-homogeneity of the unit ball of $\mathcal{L}_p(H,K)$ provided this space is not a Hilbert space.

The essential tool for this result is the following theorem proved by Stachó in [10] as a consequence of the general theory of bounded circular domains, first developed by Kaup and Upmeier in [5]. For a Banach space $F$, $\mathcal{L}_s^2(F)$ denotes the space of continuous bi-linear symmetric functions from $F \times F$ to $F$.

THEOREM. Let $F$ be a complex Banach space and $B$ its open unit ball. There exists a closed linear subspace $F_0$ of $F$ such that $(\operatorname{Aut}(B))(0) = F_0 \cap B$. Moreover, given $c \in F$, we have $c \in F_0$ if and only if there exists $Q \in \mathcal{L}_s^2(F)$ such that $\lambda(Q(a,a)) = \|a\|^2 \cdot \overline{\lambda(c)}$ whenever $a \in F$, $\lambda \in F^*$ and $\lambda(a) = \|a\| \cdot \|\lambda\|$.

Before stating our theorem we remark that $\mathcal{L}_p(H,K)$ is a Hilbert space if and only if one of the following conditions holds: *a)* $p=2$; *b)* $\dim_{\mathbf{C}} H = 1$; *c)* $\dim_{\mathbf{C}} K = 1$.

THEOREM 4. *Suppose* $1 \leq p < \infty$, $p \neq 2$ *and* $H$ *and* $K$ *are at least 2-dimensional. Every holomorphic automorphism of the unit ball of* $\mathcal{L}_p(H,K)$ *fixes the origin.*

PROOF. We must show that $\mathcal{L}_p(H,K)_0 = \{0\}$. Equivalently, given $c \in \mathcal{L}_p(H,K)$ and $Q \in \mathcal{L}_s^2(\mathcal{L}_p(H,K))$ such that $\lambda(Q(a,a)) = \|a\|^2 \cdot \overline{\lambda(c)}$ $\forall a \in \mathcal{L}_p(H,K)$, $\lambda \in \mathcal{L}_p(H,K)^*$ with $\lambda(a) = \|a\| \cdot \|\lambda\|$, we must check that $c=0$.

In order to prove that $c=0$ it is enough to show that $(c\phi_1|\psi_1)=0$ for a pair of arbitrary unit vectors $\phi_1 \in H$ and $\psi_1 \in K$.

We remark that $\phi_1 \otimes \overline{\psi_1} \in \mathcal{F}(K,H) \subset \mathcal{L}_p(H,K)^*$ and moreover

$$(\phi_1 \otimes \overline{\psi_1})(c) = \mathrm{tr}\,(c \cdot \phi_1 \otimes \overline{\psi_1}) = \mathrm{tr}\,((c\phi_1) \otimes \overline{\psi_1}) = (c\phi_1|\psi_1)\,.$$

Let $\phi_2 \in H$ and $\psi_2 \in K$ be unit vectors respectively orthogonal to $\phi_1$ and $\psi_1$ and for $\rho > 0$ and $\theta \in \boldsymbol{R}$ let $a = \psi_1 \otimes \overline{\phi_1} + \rho e^{i\theta} \psi_2 \otimes \overline{\phi_2} \in \mathcal{F}(H,K) \subset \mathcal{L}_p(H,K)$, $\lambda = \phi_1 \otimes \overline{\psi_1} + \rho^{p-1} e^{-i\theta} \phi_2 \otimes \otimes \overline{\psi_2} \in \mathcal{F}(K,H) \subset \mathcal{L}_p(H,K)^*$. We show first that $\lambda(a) = \|a\| \cdot \|\lambda\|$; in fact, if $p=1$, $\mathcal{L}_1(H,K)^* \cong \mathcal{L}(K,H)$ and $\|a\|_1 = 1+\rho$, $\|\lambda\| = 1$, $\lambda(a) = 1+\rho$ while, if $1 < p < \infty$, $\mathcal{L}_p(H,K)^* \cong \mathcal{L}_q(K,H)$ and $\|a\|_p = (1+\rho^p)^{1/p}$, $\|\lambda\|_q = (1+\rho^{q(p-1)})^{1/q} = (1+\rho^p)^{1-1/p}$, $\lambda(a) = 1+\rho^p$. It follows that $\lambda(Q(a,a)) = \|a\|^2 \cdot \overline{\lambda(c)}$, which, setting $\gamma_j = (\phi_j \otimes \overline{\psi_j})(c)$ $(j=1,2)$, $\beta_{jk}^l = (\phi_l \otimes \overline{\psi_l})(Q(\psi_j \otimes \overline{\phi_j}, \psi_k \otimes \overline{\phi_k}))$ $(j,k,l = 1,2)$, can be re-written as $(\rho^2 \beta_{22}^1) e^{2i\theta} + (2\rho\beta_{21}^1 + \rho^{p-1}\beta_{22}^2 - (1+\rho)^{2/p}\rho^{p-1}\overline{\gamma_2}) e^{i\theta} + (\beta_{11}^1 + 2\rho^p \beta_{12}^2 - (1+\rho)^{2/p}\overline{\gamma_1}) + (\rho^{p-1}\beta_{11}^2) e^{-i\theta} = 0$. This identity holds for all $\theta \in \boldsymbol{R}$ (remark that the numbers $\gamma_j$ and $\beta_{jk}^l$ are independent of $\theta$ and $\rho$), hence all the coefficients of the powers of $e^{i\theta}$ are 0; in particular

$$\beta_{11}^1 + 2\rho^p \beta_{12}^2 - (1+\rho)^{2/p}\overline{\gamma_1} = 0 \qquad \forall \rho > 0\,.$$

Dividing by $\rho^2$ and passing to the limit as $\rho \to \infty$ we obtain that

$$\lim_{\rho \to \infty} 2\beta_{12}^2 \rho^{p-2}$$

exists and equals $\overline{\gamma_1}$. Since $p \neq 2$ this limit must vanish. Hence $\gamma_1 = 0$, *i.e.* $(c\phi_1|\psi_1) = 0$ and the proof is complete. □

**4.** Since $\mathcal{L}_2(H,K)$ is a Hilbert space, its open unit ball is homogeneous (see *e.g.* [2]) and the isotropy group of the origin consists of all unitary operators. These operators can be explicitely constructed as soon as an orthonormal basis of $\mathcal{L}_2(H,K)$ is exhibited.

Since for $T$, $S \in \mathcal{L}_2(H,K)$ we have $(T|S) = \mathrm{tr}\,(S^*T)$, we can easily prove the following:

PROPOSITION 4. *If* $\{\phi_\alpha\}_{\alpha \in A}$ *and* $\{\psi_\beta\}_{\beta \in B}$ *are orthonormal bases of* $H$ *and* $K$ *respectively,*

$$\{\psi_\beta \otimes \overline{\phi_\alpha}\}_{\substack{\alpha \in A \\ \beta \in B}}$$

*is an orthonormal basis of* $\mathcal{L}_2(H,K)$.

We conclude with the description of $\mathfrak{I}(\mathcal{L}_1(H,K))$.

THEOREM 5. Every element of $\mathfrak{I}(\mathcal{L}_1(H,K))$ is the restriction of an element of $\mathfrak{I}(\mathcal{L}(H,K))$, and conversely.

PROOF. Every $j \in \mathfrak{I}(\mathcal{L}(H,K))$ can be written in one of the following forms:

*a)* $j: T \mapsto UTV$; *b)* $j: T \mapsto U\tau(T)\,V$.

Thus, for $T \in \mathcal{L}_1(H,K)$ we have respectively:

*a)* $\|j(T)\|_1 = \operatorname{tr}([j(T)]) = \operatorname{tr}(((UTV)^*(UTV))^{1/2}) =$

$$= \operatorname{tr}((V^*T^*TV)^{1/2}) = \operatorname{tr}(V^*(T^*T)^{1/2}V) = \operatorname{tr}([T]) = \|T\|_1 .$$

*b)* $\|j(T)\|_1 = \|\tau(T)\|_1 = \|T\|_1$.

It follows that the restriction of $j$ defines an element of $\mathfrak{I}(\mathcal{L}_1(H,K))$.

Conversely, suppose $j \in \mathfrak{I}(\mathcal{L}_1(H,K))$.

Since $\mathcal{L}_1(H,K)^* \cong \mathcal{L}(K,H)$, $j^*$ belongs to $\mathfrak{I}(\mathcal{L}(K,H))$, and therefore it has one of the following forms:

*a)* $j^*: S \mapsto USV$; *b)* $j^*: S \mapsto U\tau(S)\,V$.

For $T \in \mathcal{L}_1(H,K)$, $S \in \mathcal{L}(H,K)$ we have respectively:

*a)* $S(j(T)) = (j^*(S))(T) = \operatorname{tr}(j^*(S)\cdot T) = \operatorname{tr}(USVT) = \operatorname{tr}(SVTU) = S(VTU)$.

*b)* $S(j(T)) = \operatorname{tr}(U\tau(S)\,VT) = \operatorname{tr}(\tau(T)\,\tau(V)\,S\tau(U)) =$

$$= \operatorname{tr}(S\tau(U)\,\tau(T)\,\tau(V)) = S(\tau(U)\,\tau(T)\,\tau(V)).$$

It follows that, respectively, *a)* $j: T \mapsto VTU$; *b)* $j: T \mapsto \tau(U)\,\tau(T)\,\tau(V)$,

thus $j$ is the restriction of an element of $\mathfrak{I}(\mathcal{L}(H,K))$ (remark that $\tau(U)$ and $\tau$(V) are unitary operators). □

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