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Differential geometry of Cartan domains of type four

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Geometria. — *Differential geometry of Cartan Domains of type four.* Nota di
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ABSTRACT. — In this note we compute the sectional curvature for the Bergman metric of the Cartan domain of type IV and we give a classification of complex totally geodesic manifolds for this metric.

KEY WORDS: Curvature; Geodesic; Totally geodesic manifold.

RIASSUNTO. — *Geometria differenziale per domini di Cartan di tipo IV.* In questa nota si calcolano le curvature sezionali per la metrica di Bergman del dominio di Cartan di tipo IV e si trova una classificazione completa delle varietà totalmente geodetiche con spazio tangente complesso per tale metrica.

INTRODUCTION

In E. Cartan's classification, a domain of type four is biholomorphically equivalent to the bounded symmetric domain $\mathcal{D}_n = \{z \in \boldsymbol{C}^n : |z| < 1,\ 1 - 2|z|^2 + |{}^t zz|^2 > 0\}$, where $z = {}^t(z_1, \ldots, z_n)$ and the norm $|z|$ is associated to the euclidean scalar product $(u, v) = {}^t \overline{v} u = \sum u_j \overline{v}_j$, for $u, v \in \boldsymbol{C}^n$.

The main purpose of this note will be that of developing a few elementary facts of the differential geometry of invariant metrics of $\mathcal{D}_n$.

In the first section we compute the sectional curvature of the Bergman metric of $\mathcal{D}_n$ determining two bounds and investigating its planar sections, *i.e.* sections on which the sectional curvature vanishes.

In §2 we consider totally geodesic manifolds in $\mathcal{D}_n$ and exhibit a complete classification for totally geodesic manifolds which are complex.

1. BERGMAN METRIC AND CURVATURE BOUNDS

The Cartan domain $\mathcal{D}_n$ is a bounded symmetric domain, whose Bergman kernel function is $b_{\mathcal{D}_n}(z) = (1 - 2|z|^2 + |{}^t zz|^2)^{-n}$.

The group $\operatorname{Aut}\mathcal{D}_n$ of all holomorphic automorphisms of $\mathcal{D}_n$ can be described in the following way:

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n,2),\ A \in M(n,\boldsymbol{R}),\ B \in M(n,2,\boldsymbol{R}),\right.$$
$$\left. C \in M(2,n,\boldsymbol{R}),\ D \in M(2,\boldsymbol{R}),\ \det D > 0 \right\},$$

and consider the map $\Phi : G \to \operatorname{Aut}(\mathcal{D}_n)$ defined by

$$\Phi_g(z) = \left(Az + B \begin{pmatrix} (1/2)(w+1) \\ (i/2)(w-1) \end{pmatrix} \right) \cdot \left((1\ i) \left(Cz + D \begin{pmatrix} (1/2)(w+1) \\ (i/2)(w-1) \end{pmatrix} \right) \right)^{-1},$$

where $w = {}^t zz$.

(*) Nella seduta del 18 novembre 1989.

It is possible to prove that Φ is a surjective homomorphism whose kernel is $\pm I_{n+2}$.

The Shilov boundary of $\mathcal{D}_n$ is given by $S = \{z = e^{i\theta}x, \theta \in \boldsymbol{R}, x \in \boldsymbol{R}^n, |x| = 1\}$ and the isotropy group of 0, $(\operatorname{Aut}\mathcal{D}_n)_0$, is transitive on S.

Since $\mathcal{D}_n$ is homogeneous, in order to compute the sectional curvature for the Bergman metric of $\mathcal{D}_n$ it suffices to consider one particular point of $\mathcal{D}_n$. It turns out that the cartesian coordinates $z_1, \ldots, z_n$ in $\boldsymbol{C}^n$ are geodesic coordinates at 0, up to renormalization, in the sense that the coefficients $g_{j\bar{k}}(z) = \partial^2 \ln b_{\mathcal{D}_n}(z)/\partial z_j \partial \bar{z}_k$ of the Bergman metric and those of the Levi-Civita connection $\Theta^{\cdot}_{\cdot\cdot}$ are given at 0 by $g_{j\bar{k}}(0) = 2\delta_{jk}$, $\Theta^{\cdot}_{\cdot\cdot}(0) = 0$. By consequence, the Riemann curvature tensor at 0 is

$$R_{a\bar{b}c\bar{d}}(0) = -4(\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}).$$

Throughout the following u and v will denote two linearly independent vectors in $T_0(\mathcal{D}_n) \sim \boldsymbol{C}^n$. The sectional curvature of the plane section spanned by u and v is

$$K(u,v) = (-2\operatorname{Re}({}^t uu\,\overline{{}^t vv}) + 2|(u,\bar{v})|^2 + (u,v)^2 + \overline{(u,v)}^2 - \\ -2|u|^2|v|^2 - 4\operatorname{Im}^2(u,v))(4|u|^2|v|^2 - (u,v)^2 - \overline{(u,v)}^2 - 2|(u,v)|^2)^{-1},$$

where (u, v) is the standard inner product in $\boldsymbol{C}^n$ and $|u|^2 = (u, u)$.

As $K(u, v)$ does not depend on the choice of the two vectors in the plane spanned by u and v, we can suppose that $\operatorname{Re}(g_{j\bar{k}}u_j\bar{v}_k) = \operatorname{Re}((u,v)) = 0$.

We deduce from that $(u,v)^2 + \overline{(u,v)}^2 + 2|(u,v)|^2 = 0$, hence

$$K(u,v) = (-\operatorname{Re}({}^t uu\,\overline{{}^t vv}) - |u|^2|v|^2 + |(u,\bar{v})|^2 + 3(u,v)^2)(2|u|^2|v|^2)^{-1}.$$

Since $|\operatorname{Re}({}^t uu\,\overline{{}^t vv})| < |u|^2|v|^2$, $|(u,\bar{v})|^2 < |u|^2|v|^2$ and $(u,v)^2 \leqslant 0$, the sectional curvature is bounded by $-5/2 \leqslant K(u,v) \leqslant 1/2$.

These estimates might possibly be improved. An indication in this direction is given by the fact that the bounds just found cannot be reached: $K(u,v) = -5/2$ implies $(u,v) = -|u|\,|v|$, so $u \in \boldsymbol{C}v$, that is impossible. As for the upper bound, note that, if $K(u,v) = 1/2$, then $\operatorname{Re}({}^t uu\,\overline{{}^t vv}) = -|u|^2|v|^2, (u,v) = 0$ and $|(u,\bar{v})| = |u||v|$; from this we deduce that $\bar{v} = e^{i\theta}u$, then $\operatorname{Re}((u,\bar{u})(\bar{v},v)) = \operatorname{Re}(u, e^{i\theta}v)(\bar{v},v) = 0$, showing that 1/2 is not reached.

We characterize now the planar sections, *i.e.* plane sections determined by u and v on which $K(u, v) = 0$. To find such sections first we fix u; then we find v such that $K(u,v) = 0$, assuming of course $|u| = |v| = 1$ (notice that the square of the length of a vector for the Bergman metric in 0 is twice the square of its length for the euclidean norm). Hirzebruch proved in [5] that.

THEOREM 1.1. For all $x \in \boldsymbol{C}^n$ there is $A \in O(n) \subset (\operatorname{Aut}\mathcal{D}_n)_0$ such that $Ax = e^{i\theta}\cdot{}^t(a, ib, 0, \ldots, 0)$, with $a, b \in \boldsymbol{R}$.

Then, setting $N(u) = \{v \in \boldsymbol{C}^n : K(u,v) = 0,\ \operatorname{Re}(u,v) = 0\}$, it is easily seen that, if $A \in O(n)$, then $v \in N(u) \Leftrightarrow Av \in N(Au)$, hence we can suppose that $u = e^{i\theta}\cdot{}^t(\cos r, i\sin r, 0, \ldots, 0)$.

Since $v \in N(u) \Leftrightarrow e^{i\theta}v \in N(e^{i\theta}u)$, we can assume that $u = {}^t(\cos r, i\sin r, 0, \ldots, 0)$.

For $v = {}^t(z_1, z_2, \ldots, z_n) = {}^t(x_1 + iy_1, \ldots, x_n + iy_n)$, with x_j, y_j in $\boldsymbol{R}$ we have two distinct cases:

a) If $\sin r = 0$, *i.e.* $u = {}^t(1, 0, \ldots, 0)$, $\operatorname{Re}(u, v) = 0$ implies $x_1 = 0$. Hence

$$0 \leqslant 1 + \operatorname{Re}{}^t vv = |z_1|^2 + 3z_1^2 = -2y_1^2 \leqslant 0 \Rightarrow y_1 = 0 \quad \text{and} \quad \operatorname{Re}{}^t vv = -1,$$

and therefore v must have the form $v = {}^t(0, iy_2, \ldots, iy_n)$, where $\sum_{j=2}^{n} y_j^2 = 1$.

b) If $\sin r \neq 0$, $\operatorname{Re}(u, v) = 0$ implies $y_2 = -\cot r x_1$ and ${}^t uu = (\cos^2 r - \sin^2 r)$, $\operatorname{Re}{}^t vv = x_1^2 - y_1^2 + \ldots + x_n^2 - y_n^2 = 2(x_1^2 + \ldots + x_n^2) - 1$.

Thus, by setting $s = x_3^2 + \ldots + x_n^2$, $K(u, v) = 0 \Leftrightarrow 1 + \cos 2r\,(2x_1^2 + 2x_2^2 + 2s - 1) = 4x_1^2 \cos^2 r - 2x_2^2 \sin^2 r - 2y_1^2 \cos^2 r + 8x_2 y_1 \sin r \cos r$, *i.e.*

$$1 + 2(x_2^2 \cos^2 r + y_1^2 \cos^2 r - 4x_2 y_1 \cos r \sin r) + \cos 2r(2s - 1) = 2x_1^2. \tag{1}$$

That proves the following proposition which yields all planar sections determined by u and v in $\boldsymbol{C}^n$.

PROPOSITION 1.2. The unitary vectors u and v in $\boldsymbol{C}^n$ determine a planar section if and only if there exists an element $\varphi \in (\operatorname{Aut}\mathcal{D}_n)_0$ such that either

i) $\varphi u = {}^t(1, 0, \ldots, 0)$ and $\varphi v = {}^t(0, iy_2, \ldots, iy_n)$, or ii) $\varphi u = {}^t(\cos r, i \sin r, 0, \ldots, 0)$ and $\varphi v = (x_1 + iy_1, x_2 i + i \cot r x_1, \ldots, x_n + iy_n)$, where φv satisfies (1).

We shall now compute the holomorphic sectional curvature determined by u in $\boldsymbol{C}^n$, that is the curvature of the plane section determined by u and $v = iu$.

Since $\operatorname{Re}(u, v) = 0$, then $K(u, iu) = (|{}^t uu|^2 - 2|u|^4)|u|^{-4} = |{}^t uu|^2 |u|^{-4} - 2$.

Hence the bounds for the holomorphic sectional curvature are -2 and -1:

$$K(u, iu) = -2 \Leftrightarrow {}^t uu = 0 \Leftrightarrow (u, \overline{u}) = 0,$$

$$K(u, iu) = -1 \Leftrightarrow |{}^t uu| = |u|^2 \Leftrightarrow |(u, \overline{u})| = (u, u) \Leftrightarrow u = e^{i\theta} x,$$

where $x \in S^{n-1}$ the unit sphere in $\boldsymbol{R}^n$.

The holomorphic bisectional curvature at $0 \in \mathcal{D}_n$ along the complex plane spanned by u and v is given by

$$\begin{aligned} K_b(u, v) &= -(R_{a\bar{b}c\bar{d}} u^a \overline{u}^b v^c \overline{v}^d)(g_{a\bar{b}} g_{c\bar{d}} u^a \overline{u}^b v^c \overline{v}^d)^{-1} = \\ &= 4(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd})\, u^a \overline{u}^b v^c \overline{v}^d (4|u|^2|v|^2)^{-1} = \\ &= (-|u|^2|v|^2 - |(u, v)|^2 + |(u, \overline{v})|^2)(|u|^2|v|^2)^{-1}. \end{aligned}$$

First of all that implies $K_b(u, v) = K(u, iu)$ if $v \in \boldsymbol{C}u$.

The bounds of K_b are -2 and 0 and they turn out to be the best possible: in fact $K_b(u, v) = -2$ if and only if $|(u, \overline{v})| = 0$ and $|(u, v)| = |u||v|$, that is, if and only if, $v = e^{i\theta} u$, with $(u, \overline{u}) = 0$. In particular v must lie in the complex line determined by u, and therefore $K_b(u, v) = K(u, iu)$. As for the lower bound, note that $K_b(u, v) = 0 \Leftrightarrow (u, v) = 0$ and $(u, \overline{v}) = |u||v|$, *i.e.* $(u, \overline{u}) = 0$ and $v = e^{i\theta} \overline{u}$.

The results can be summarized as follows

PROPOSITION 1.3. The bounds for the holomorphic sectional curvature $K(u, iu)$ are -2 and -1: the first is reached if and only if $(u, \overline{u}) = 0$, the second if and only if $u = e^{i\theta} x$, where $x \in S^{n-1} \subset \boldsymbol{R}^n$. The bounds for the holomorphic bisectional curvature $K_b(u, v)$ are -2 and 0: the first is reached if and only if v is in $\boldsymbol{C}u$ and $(u, \overline{u}) = 0$, the second if and only if $(u, \overline{u}) = 0$ and $v = e^{i\theta} \overline{u}$.

2. GEODESICS AND TOTALLY GEODESIC SUBMANIFOLDS

The first part of the following theorem has been proved by Köcher in [8] and Hirzebruch in [5] (see also [3], where the proof has been considerably simplified); the description of the geodesics for the Bergman metric of $\mathcal{D}_n$ follows from simple considerations on the proof in [3].

THEOREM 2.1. Let z_1 and z_2 in $\mathcal{D}_n$, there is a unique geodesic for the Bergman metric φ such that $\varphi(0) = z_1$ and $\varphi(1) = z_2$. Such a geodesic is obtained as the image, by a suitable automorphism of $\mathcal{D}_n$, of the curve $\varphi_1(t) = ((\tanh tx + \tanh ty) 2^{-1}, (\tanh tx - \tanh ty)(2i)^{-1}, 0, \dots, 0)$, where $t \in \boldsymbol{R}$, $x, y \in \boldsymbol{R}$.

This implies that $\mathcal{D}_2$ is totally geodesic in $\mathcal{D}_n$.

Now we want to study the totally geodesic manifolds in $\mathcal{D}_n$ whose tangent spaces are complex subspaces of $\boldsymbol{C}^n$. From now on we shall indicate them as C.T.G.M.

The domain $\mathcal{D}_n$ being homogeneous, we can limit ourselves to the costruction of a C.T.G.M. W with $0 \in W$.

PROPOSITION 2.2. The C.T.G.M. in $\mathcal{D}_n$ of complex dimension 1 are

$$A_1 = \{z \in \mathcal{D}_n : z = {}^t(z_1, 0, \dots, 0)\}, \quad A_2 = \{z \in \mathcal{D}_n : z = {}^t(z_1, -iz_1, 0, \dots, 0)\},$$

and all their images under automorphisms of $\mathcal{D}_n$.

PROOF. Consider w_1 in $T_0(W)$, the tangent space of W in 0, because of Theorem 1.1 we can suppose that $w_1 = {}^t(x_1, ix_2, 0, \dots, 0)$, with $x_1, x_2 \in \boldsymbol{R}$, $x_1^2 + x_2^2 > 0$.

It is easy to check that the restriction to $\mathcal{D}_2$ of the linear map $\tau = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ gives a biholomorphism between $\mathcal{D}_2$ and $\Delta \times \Delta$.

Then we can study C.T.G.M. in $\Delta \times \Delta$: the geodesic in $\Delta \times \Delta$ whose tangent vector in 0 is $(x_1 - x_2, x_1 + x_2)$ is $\gamma(t) = (\tanh(t(x_1 - x_2)), \tanh(t(x_1 + x_2)))$.

Also $i\gamma(\boldsymbol{R})$ is in $\tau(W)$, because its tangent vector in 0 is $i(x_1 - x_2, x_1 + x_2)$. Let $P = \gamma(1) = (r_1, r_2)$ and $Q = iP$. If ψ is the geodesic such that $\psi(0) = P$ and $\psi(1) = Q$, then $\psi(\boldsymbol{R})$ is contained in $\tau(W)$, therefore $\psi(-1)$ is in $\tau(W)$. Then the geodesic ν such that $\nu(1) = \psi(-1)$ and $\nu(0) = 0$ must have tangent vector in $\boldsymbol{C}(x_1 - x_2, x_1 + x_2)$.

This can happen if and only if either

$$\text{i) } \tau(W) = \{z \in \Delta \times \Delta : z = {}^t(z_1, z_1)\}, \quad \text{or} \quad \text{ii) } \tau(W) = \{z \in \Delta \times \Delta : z = {}^t(0, z_2)\}.$$

In fact $\psi(t) = (\gamma_1(t), \gamma_2(t))$, where γ_j is the geodesic such that $\gamma_j(0) = r_j$ and $\gamma_j(1) = ir_j, j = 1, 2$. Since $\nu(1) = {}^t((2r_1 - ir_1 - ir_1^3) \cdot (1 - 2ir_1^2 + r_1^2)^{-1}, (2r_2 - ir_2 - ir_2^3) \cdot (1 - 2ir_2^2 + r_2^2)^{-1})$ setting $\nu(1) = {}^t(e^{i\theta_1} \tanh a, e^{i\theta_2} \tanh b)$, the tangent vector to ν in 0 is ${}^t(e^{i\theta_1} a, e^{i\theta_2} b)$. This vector is in $\boldsymbol{C}(x_1 - x_2, x_1 + x_2)$ if and only if either $\theta_1 = \theta_2 + k\pi$, for some integer k or $ab = 0$. Thus we have the two manifolds of cases i) and ii). Applying τ^{-1} to these manifolds we obtain the thesis. □

We pass now to the k-dimensional case proving

THEOREM 2.3. The C.T.G.M. in $\mathcal{D}_n$ are obtained as images by $\operatorname{Aut}\mathcal{D}_n$ of either

1) $M_1 = \{z \in \mathcal{D}_n : z = {}^t(z_1, \dots, z_k, 0, \dots, 0)\}$ and

2) $M_2 = \{z \in \mathcal{D}_n : z = {}^t(z_1, iz_1, \dots, z_{2k-1}, iz_{2k-1}, 0, \dots, 0)\}$.

We need the following

LEMMA 2.4. $W = \{z \in \mathcal{D}_3 : z = {}^t(z_1, z_2, iz_2)\}$ is not a C.T.G.M.

PROOF. The vectors $w_1 = {}^t(a, 0, 0)$ and $w_2 = {}^t(0, b, ib)$, where $a, b \in \boldsymbol{R}$, are a complex base for $T_0(W)$, the tangent space of W in 0.

Let $P = {}^t(\operatorname{tgh} a, 0, 0)$ and $Q = {}^t(0, (\operatorname{tgh} 2b)\, 2^{-1}, i(\operatorname{tgh} 2b)\, 2^{-1})$ and let γ be the geodesic such that $P = \gamma(0)$ and $Q = \gamma(1)$: if $\gamma(-1) = {}^t(v_1, v_2, v_3)$ and $v_2 + iv_3 \neq 0$ then W is not a C.T.G.M.

Let $x = \tanh a$ and $y = \tanh b$; with a brief calculation we have that

$$\gamma(-1) = d^{-1} \cdot (1 - 4y^2)^{-1} \begin{bmatrix} * \\ iy(x^2 + 1) - iy(w'^2 - 1) \\ y(x^2 + 1) - y(w'^2 + 1) \end{bmatrix},$$

where $w'^2 = x^4$ and d is a constant factor. The condition $v_2 + iv_3 = 0$ is not possible, so W is not a C.T.G.M. □

From now on e_j will denote the j-th element of the standard base in $\boldsymbol{C}^n$. Then we have the following

COROLLARY 2.5. If $e_1, e_2 + ie_3 \in T_0(W)$, where W is a C.T.G.M., then $e_2, e_3 \in T_0(W)$.

PROOF (of the theorem). We prove Theorem 2.3 in two steps. First we prove that M_1 and M_2 are C.T.G.M., then we show that M_1, M_2 and all their images by elements in $\operatorname{Aut}\mathcal{D}_n$ are the only possible C.T.G.M.

To prove the first part of thesis, it is enough to show that M_1 and M_2 are C.T.G.M. It suffices to show that the two subgroups of $\operatorname{Aut}\mathcal{D}_n$ leaving M_1 and M_2 invariant act transitively on M_1 and M_2 respectively, and that these manifolds are totally geodesic in 0.

For $M_1 = \mathcal{D}_k \times \{0\}^{n-k}$ both statements are trivial. For M_2 the proof is a bit more difficult.

First of all we prove that M_2 is totally geodesic in 0. Set $w = {}^t(z_1, iz_1, \dots, z_{2k-1}, iz_{2k-1}, 0, \dots, 0)$, and choose $L = (l_{jk})$ in $O(k)$ such that $L^t(z_1, z_3, \dots, z_{2k-1}) = {}^t(x, iy, 0, \dots, 0)$. Consider

$$B = \begin{bmatrix} l_{11} & 0 & l_{12} & 0 & \dots \\ 0 & l_{11} & 0 & l_{12} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix};$$

it is evident that $Bw = {}^t(x, ix, iy, -y, 0, \dots, 0)$.

Let us define $n = (x^2 + y^2)^{-1/2}$ and

$$F = \begin{bmatrix} xn & 0 & 0 & -yn & 0 \\ 0 & xn & yn & 0 & 0 \\ 0 & -yn & xn & 0 & 0 \\ yn & 0 & 0 & xn & 0 \\ 0 & 0 & 0 & 0 & I_{n-4} \end{bmatrix}.$$

Both B and F trasform M_2 onto itself and $FBw = (n, in, 0, \dots, 0)$. The fact that in $\mathcal{D}_2$ $M = \{z \in \mathcal{D}_2 : z = {}^t(z_1, iz_1)\}$ is totally geodesic in 0, implies that M_2 is totally geodesic in 0.

To see that M_2 is homogeneous under restrictions of automorphisms of $\mathcal{D}_n$ it is enough to check that, given $z_0 \neq 0$, in M_2, there exists a matrix g_{z_0} in G such that $\Phi_{g_{z_0}}(z_0) = 0$ and $\Phi_{g_{z_0}}(M_2) \subset M_2$.

With the notations of §1,

$$g_{z_0} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = I + a|z_0|^{-2}(z_0^t \bar{z}_0 + \bar{z}_0^t z_0)$, $\quad a = (1 - |z_0|^2)^{-2^{-1}} - 1$,

$$D = (1 - 2|z_0|^2)^{-2^{-1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = AX_0 \qquad \text{and} \qquad C = D^t X_0,$$

where $X_0 = (2(z_0 - i\bar{z}_0))(2(z_0 + i\bar{z}_0))$(for a proof of the fact that $g_{z_0} \in G$ and $\Phi_{g_{z_0}}(z_0) = 0$ see [6]).

Then $\Phi_{g_{z_0}}(z) = d^{-1}(Az - Az_0)$, where d is a constant, for all z in M_2, because ${}^t zz = 0$. Since $\Phi_{g_{z_0}}(z) \in \mathcal{D}_n$, what we are left to prove is that $\Phi_{g_{z_0}}(z) \in V = \{z \in C^n : z = {}^t(z_1, iz_1, \dots, z_{2k-1}, iz_{2k-1}, 0, \dots, 0)\}$, as V is a vector space, $(z_0^t \bar{z}_0 + \bar{z}_0^t z_0)(z - z_0)$ is in V iff $\bar{z}_0^t z_0(z - z_0) \in V$: if we show that this is 0 we have that M_2 is a C.T.G.M. If we choose u and v in V we have $u = {}^t(u_1, iu_1, \dots, u_{2k-1}, iu_{2k-1}, 0, \dots, 0)$ and $v = {}^t(v_1, iv_1, \dots, v_{2k-1}, iv_{2k-1}, 0, \dots, 0)$ then ${}^t uv = u_1v_1 + i^2u_1v_1 + \dots + u_{2k-1}v_{2k-1} + i^2u_{2k-1}v_{2k-1} = 0$, hence $\bar{z}_0^t z_0(z - z_0) = 0$, and we have proved the first part of the thesis.

We now come to the second step of the proof of Theorem 2.3. Let W be a C.T.G.M. such that $0 \in W$. Note that, if $w \in T_0(W) \sim C^n$, there are three possibilities:

$$\text{i) } w \in \mathcal{S}, \qquad \text{ii) } {}^t ww = 0, \qquad \text{iii) } w \notin \mathcal{S}, \quad \text{and} \quad {}^t ww \neq 0$$

and these possibilities are preserved by the action of $(\mathrm{Aut}\,\mathcal{D}_n)_0$.

We fix an orthonormal base $w_1, \dots, w_k$ of $T_0(W)$ containing the maximum number of elements which satisfy either i) or iii).

Rearranging the base we can suppose that $w_1, \dots, w_r$ satisfy i), $w_{r+1}, \dots, w_s$ satisfy ii) and $w_{s+1}, \dots, w_k$ satisfy iii). Note that, if we multiply each w_j for a constant of modulus 1, the base we obtain has still the same properties.

Applying a suitable element $A \in O(n)$ to w_1 we obtain $Aw_1 = e^{i\theta t}(1, 0, \dots, 0)$; as we consider W modulus the action of $(\mathrm{Aut}\,\mathcal{D}_n)_0$ we can suppose that $w_1 = {}^t(1, 0, \dots, 0)$, then

$w_2, \dots, w_k$ have the first coordinate equal to 0. Repeating this method acting only on the last non vanishing coordinates we can suppose that $w_j = e_j$ for $j = 1, \dots, r$, and w_h has the first r coordinates equal to 0 for $h = r+1, \dots, k$.

Applying a suitable element of $O(n)$ that is the identity on the first r coordinates we can suppose that $w_{r+1} = {}^t(0, \dots, 0, x, ix, 0, \dots, 0)$, where $x \in \boldsymbol{R} - \{0\}$.

If $j = r+2, \dots, k$ and $w_j = {}^t(0, \dots, 0, z_{r+1}, \dots, z_n)$ then $z_{r+1} = iz_{r+2}$, because the base is orthogonal.

For each fixed $h \in \{r+2, \dots k\}$ consider the unitary map of $\boldsymbol{C}^k$ defined by

$$w_{r+1} \mapsto w'_{r+1} = \cos\theta w_{r+1} + \sin\theta w_h ,$$

$$w_h \mapsto w'_h = -\sin\theta w_{r+1} + \cos\theta w_h ,$$

$$w_m \mapsto w_m , \qquad \text{if } m \neq r+1,\ h .$$

As ${}^t w'_{r+1} w'_{r+1} = 2\cos\theta\sin\theta\, {}^t w_{r+1} w_h + \sin^2\theta\, {}^t w_h w_h$ and ${}^t w'_h w'_h = -2\cos\theta\sin\theta\, {}^t w_{r+1} w_h + \cos^2\theta\, {}^t w_h w_h$, then, if ${}^t w_{r+1} w_h \neq 0$, there is a suitable θ for which ${}^t w'_{r+1} w'_{r+1} \neq 0$ and ${}^t w'_h w'_h \neq 0$. So we can replace w_{r+1} and w_h by w'_{r+1} and w'_h none of which satisfies ii), this is absurd because of the choice of the base; then ${}^t w_{r+1} w_h = 0$ for all $h = r+2, \dots, k$.

Hence we obtain that $w_h = {}^t(0, \dots, 0, z_{r+3}, \dots, z_n)$, for $h \in \{r+2, \dots, k\}$. We continue by the same method acting only on the last $n - (r+2)$ coordinates and we end up with $T_0(W)$ containing $e_{r+1} + ie_{r+2}, \dots, e_{2s-r-1} + ie_{2s-r}$ and with $w_{s+1}, \dots, w_k$ having the first $2s - r$ coordinates equal to 0.

Choosing a suitable element of $O(n)$ which is the identity map on the first $2s - r$ coordinates and applying it to w_{s+1}, we can suppose that $w_{s+1} = ae_{2s-r+1} + ibe_{2s-r+2}$, where $a, b \in \boldsymbol{R}$ and $a \neq b$. Hence, by Proposition 2.2, e_{2s-r+1} and $e_{2s-r+2} \in T_0(W)$. We want to add to e_{2s-r+1}, e_{2s-r+2} other elements so as to have an orthonormal base $e_{2s-r+1}, e_{2s-r+2}, w'_{s+3}, \dots, w'_k$ of the vector space spanned by $w_{s+1}, \dots, w_k$ in which $w'_{s+3}, \dots, w'_k$ are all in the Shilov boundary. Once this has been done, applying a suitable element of $O(n)$ which is the identity map on the first $2s - r + 2$ coordinates, we can suppose that $w'_{s+3}, \dots, w'_k$ are replaced by $e_{2s-r+3}, \dots, e_{k+(s-r)}$.

If $k - (s+r) = 2$ we have such a base already. If that is not the case we can find w'_{s+3} in the vector space spanned by $w_{s+1}, \dots, w_k$ which is orthogonal to e_{2s-r+1} and e_{2s-r+2}. Then we can suppose, applying a suitable element in $O(n)$ which is the identity map on the first $2s - r + 2$ coordinates, that $w'_{s+3} = ce_{2s-r+3} + ide_{2s-r+4}$, where $c, d \in \boldsymbol{R}$.

If w'_{s+3} is in the Shilov boundary we can go to w'_{s+4}. If ${}^t w'_{s+3} w'_{s+3} = 0$ we can apply Corollary 2.5 to w'_{s+3} and e_{2s-r+1}. Since e_{2s-r+3} and e_{2s-r+4} are now in $T_0(W)$, so we can take $w'_{s+3} = e_{2s-r+3}$ and $w'_{s+4} = e_{2s-r+4}$, and we can go on adding w'_{s+5}. If either w'_{s+3} is not in the Shilov boundary or ${}^t w'_{s+3} w'_{s+3} \neq 0$. Proposition 2.2 implies that e_{2s-r+3} and e_{2s-r+4} are in $T_0(W)$. Then we can go on adding w'_{s+5}.

In conclusion we have found a base of elements in the Shilov boundary for the complex vector space spanned by $w_{s+1}, \dots, w_k$, then, up to the action of $O(n)$, we can suppose that $T_0(W)$ is spanned by $e_1, \dots, e_j, e_{j+1} + ie_{j+2}, \dots, e_{2m+j-1} + ie_{2m+j}$, where $k = j + m$.

Applying again Corollary 2.5 to e_1 and $e_{j+1} + ie_{j+2}$ we obtain that either $j = 0$ or

$m=0$, *i.e.* $T_0(W)$ is spanned by $e_1, \dots, e_k$ (which corresponds to M_1) or by $e_1+ie_2, \dots, e_{2k-1}+ie_{2k}$ (which corresponds to M_2).

That proves that M_1, M_2 and all their images by elements of $\mathrm{Aut}\,\mathcal{D}_n$ exhaust all C.T.G.M. $\square$

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