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## On factorisable soluble groups

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ABSTRACT. — The intention of this paper is to provide an elementary proof of the following known results: Let $G$ be a finite group of the form $G = AB$. If $A$ is abelian and $B$ has a nilpotent subgroup of index at most 2, then $G$ is soluble.

KEY WORDS: Finite group; Soluble group; Factorisable group.

RIASSUNTO. — *Semigruppi risolubili fattorizzabili.* Lo scopo di questa nota è di fornire una dimostrazione elementare del seguente teorema: Sia $G$ un gruppo finito nella forma $G = AB$. Se $A$ è abeliano e $B$ ha un sottogruppo nilpotente di indice al più 2, allora $G$ è risolubile.

## INTRODUCTION

In [1] the following theorem has been proved:

THEOREM. Suppose $G = AB$ where $G$ is a finite group. If $A$ is abelian and $B$ has a nilpotent subgroup of index at most 2, then $G$ is soluble.

However, in proving the above theorem, the author uses the deep results of Gorenstein and Walter [2]. In this short note we provide a very elementary proof of the above theorem, using only results fully proved in [3] and [4]. The notation used is standard and may be found in [3] or [4].

The case where $A$ has even order has been proved in [1] without employing [2]. Thus in proving the above theorem, we shall assume that $A$ has odd order.

PROOF. The proof shall be broken in several lemmas. If $N$ is a normal subgroup of $G = AB$, $(|A|, |B|) = 1$, then a simple induction on $|G|$ shows that $N$ is factorisable (in $AN$) in $G$. Let $G$ be a minimal counterexample to the above theorem. If $M$ is a proper subgroup of $G$ containing $B$, then $H = (M \cap A)^G \subseteq M$. Since $M$ is soluble (by the minimality of $G$), $H$ is soluble. Hence $H = 1$, $B$ is a maximal subgroup of $G$ and $(|A|, |B|) = 1$. Thus $G$ is simple. Since $1 \neq Z(S) \cap O_2(B) \subseteq Z(B)$, $S \in \mathrm{Syl}_2(B)$, by means of a theorem by Burnside [4, p. 334], A is not primary. We have thus proved all parts of the following lemma.

LEMMA 1. $(|A|, |B|) = 1$, $G$ is simple and $B$ is a maximal subgroup of $G$. Further, $A$ is not primary.

LEMMA 2. A is a T.I. subgroup.

PROOF. If $D = A \cap A^g \neq 1$, then $K = N_G(D) \supseteq \langle A, A^g \rangle$. Since $K$ is soluble and $G$ is simple $F(K) \subseteq A$. Thus $C_K(F(K)) \subseteq F(K)$ gives $A = F(K) = A^g$.

LEMMA 3.

i) If $1 \neq C \subseteq A$, $1 \neq D \in \mathrm{Syl}_p(O(B))$, then $\langle C, D \rangle = G$.

(*) Nella seduta del 18 novembre 1989.

ii) If $1 \neq H \lhd F(B)$, then $N_G(H) \subseteq B$.

PROOF. We may assume that $C$ is a $q$-group for some prime $q$. If $\langle C, D\rangle \subset G$, then $N = N_G\langle C, D\rangle$ is factorisable and so is soluble. Thus $DQ^g$, with $Q \in \mathrm{Syl}_q(N)$, is a proper subgroup of $G$ for all $g \in G$. By a theorem of Kegel [4, p. 382] $G$ is not simple, contrary to lemma 1. This proves (i).

If $1 \neq H \lhd F(B)$, then $F(B) \subseteq N_G(H)$. Applying (i), $N_G(H)$ is a $\pi(B)$-group. Thus $[N_G(H) : F(B)] \leq 2$ and so $N_G(H) \subseteq N_G(F(B)) = B$.

LEMMA 4: $O_2(B)$ is a T.I. subgroup.

PROOF. Deny and choose $a \in A^{\#}$ such that $D = O_2(B) \cap O_2(B^a)$ has maximal order. Set $N = N_G(D)$. Since $O(B) \subseteq N$, lemma 3 implies $N$ is a $\pi(B)$-group. Since $[\mathrm{N} : O(B)]$ is a power of 2, $N$ is soluble by a theorem of Wielandt [4, p. 379].

If $O(N) \neq 1$, then $O(N) \lhd F(B)$ and so $N \subseteq B$ by lemma 3. Thus $O(B^a) \subseteq B$ giving $a \in B$, a contradiction. Thus $O(N) = 1$. If $D^* = O_2(N)$, then $N \subseteq N_G(D^*)$ and it is clear that $C_G(D^*) \subseteq D^*$. In particular, if $T_1$, $T_2$ are $S_2$-subgroups of $G$ containing $D^*$, then $T_1$ and $T_2$ do not lie in the same conjugate of $B$.

Now if $K$ is a Hall $2'$-subgroup of $O_{2,2'}(N)$, $K \subseteq O(B)$, then $L \subseteq C_N(K) \subseteq O_{2,2'}(N)$ where $L = O_2(B) \cap N$ *i.e.* $L \subseteq D^*$.

If $D \lhd F(B)$, the lemma follows from lemma 3. Hence an $S_2$-subgroup of $G$ is non-abelian. Now if $D^*$ lies in a unique $S_2$-subgroup of $G$, $T$ say, then $\langle O(B), O(B^a)\rangle \subseteq$ $\subseteq N_G(T) \subseteq N_G(T') = B^e$, for some $e \in A$. Again we have $a \in B$, a contradiction.

Thus $D^* \subseteq T_1 \cap T_2$ where $T_1$, $T_2$ are $S_2$-subgroups of $G$ lying in distinct conjugates $B_1$, $B_2$ of $B$. Since $[T_1 : O_2(B_1)] = 2$, we have $[T_1 \cap T_2 : O_2(B_1) \cap O_2(B_2)] \leq 4$ *i.e.* $[D^* : D] \leq 4$ and so $[D^* : L] \leq 2$. Since $O(B)$ centralises $L$ and normalises $D^*$, it follows that $O(B)$ centralises $D^*$ contrary to $C_G(D^*) \subseteq D^*$.

LEMMA 5. If $1 \neq H \subseteq O(B)$, then $N_G(H)$ is a $\pi(B)$-group.

PROOF. Deny. Since $O(B)$ is nilpotent, we may assume $H$ is a $p$-subgroup of $P$, where $P \in \mathrm{Syl}_p(B)$. Choose an $H$ of maximal order such that $A \cap N_G(H)$ contains a non-trivial $S_r$-subgroup $R$ of $N_G(H)$. Then $\langle R, O_2(B)\rangle \subseteq N$, $N = N_G(H)$. Since $N_G\langle R, O_2(B)\rangle$ is soluble, $K = DS$ is a Hall $\{2, r\}$-subgroup of $N_G\langle R, O_2(B)\rangle$, $S \supseteq O_2(B)$. If $O_r(K) \neq 1$, then $N_G(O_r(K)) \supseteq \langle A, O_2(B)\rangle$ contrary to the simplicity of $G$. Thus $O_r(K) = 1 \neq O_2(K)$. If $O_2(K) \subseteq O_2(B)$ then $N_G(O_2(K)) \supseteq \langle D, O(B)\rangle = G$ by lemma 3, another contradiction. Hence, $O_2(K) \not\subseteq O_2(B)$, $S$ is an $S_2$-subgroup of $G$, $[O_2(K) : O_2(K) \cap O_2(B)] = 2$ and $[O_2(K) : O_2(K) \cap O_2(B^x)] = 2$ for all $x \in D$. By lemma 4, $|O_2(K)| \leq 4$. Since $C_K(O_2(K)) \subseteq$ $\subseteq O_2(K)$, $|O_2(K)| = 4$ and $|S| \leq 8$ is dihedral. Thus $G$ has a unique class of involutions ([3, p. 262]) and hence $N_G(D)$ has odd order (otherwise $N_G(D) \supseteq \langle A, u\rangle$ where $u$ is a central involution in $B$). It follows from a theorem by Burnside ([4, p. 137]) that $S \lhd K$. If $|S| = 8$, then Aut $(S)$ is a 2-group and so $K = SXD$ contrary to $N_G(D)$ has odd order. Thus $|S| = 4$ and since $O_r(K) = 1$, $|D| = 3 = |R|$ and $N = N_G(H)$ has a normal Hall subgroup of index 3. By the Frattini argument, a conjugate of $R$ in $N$ normalises $P^*$, where $P^*$ is an $S_p$-subgroup of $N$ containing $N_P(H)$. Maximality of $H$ now forces $H = P$ giving $R \subseteq N_G(P) =$ $= B$, a contradiction.

LEMMA 6. $O(B)$ is a T.I. subgroup.

PROOF. Deny. If $O(B)=P$ is an $S_p$-subgroup of $G$, then choose $g \in G-B$ such that $D=P \cap P^g$ has maximal order. Hence $R=N_P(D)$, $U=N_{P^g}(D)$ are $S_p$-subgroups of $N_G(D)$. By lemma 5, $[N_G(D):O_2(B)R] \leqslant 2$. Since $O_2(B)R=O_2(B)XR$, $R=U$, a contradiction.

We may assume $\pi(O(B)) \supseteq \{p,q\}$, $p \neq q$. We first assert that if $\pi(K \cap Z(O(B)))=\pi_0$ for any subgroup $K$ of $G$, then $K \cap O(B)$ contains a Hall $\pi_0$-subgroup of $K$. For if $Q_0$ is a $q$-subgroup of $G$ such that $Q_0 \cap Z(Q) \neq 1$, $Q \in \mathrm{Syl}_q(B)$, then by lemma 3, $C_G\langle t\rangle \subseteq B$ where $t \in Q_0^{\#} \cap Z(Q)$. If $P$ is an $S_p$-subgroup of $B$, $P^*$ an $S_p$-subgroup of $G$ centralised by $Q_0$, then $P$, $P^* \subseteq B$ and so $P=P^*$. Now $Q_0 \subseteq N_G(P)=B$. The assertion follows.

Now assume $|\pi(B)| \geqslant 3$ and let $1 \neq D=O(B) \cap O(B^g)$ be of maximal order, $g \in G-B$. Then $Z(O(B))$, $O_2(B) \subseteq N_G(D)$ and so by lemma 5 and the assertion above $[N_G(D):B \cap N_G(D)] \leqslant 2$. Thus $O(N_G(D))=O(B) \cap N_G(D)=O(B^g) \cap N_G(D)=D$, a contradiction.

LEMMA 7. $G$ does not exist.

PROOF. Let $|A|=a$, $|F(B)|=h$ and $|N|=ar$ where $N=N_G(A)$. Let $U_1=G-B$, $U_2=G-N$ and $U_3=A^xN$, $A^x \neq A$. By lemmas 2, 4 and 6, both $F(B)$ and $A$ are T.I. subgroups of $G$. Hence, on considering the double coset decomposition of $G$ one time by $F(B)$ and $F(B)$ and another time by $A$ and $A$ we get: $|U_1|=kh^2$, $|U_2|=la^2$, $k,l \geqslant 1$. Further, $|U_3|=a^2r$.

If $a>h$, then $|U_2|<|G|$ implies $la^2<2ah<2a^2$ *i.e.* $l=1$. Thus $|G|=2ah=a(r+a)<$ $<a(ar)=|U_3|$, a contradiction.

If $h>a$, then $|U_1|<|G|$ gives $k=1$ and $|G|=2ah=2h+h^2$ *i.e.* $h=2(a-1)$. Also $|U_3|<|G|$ implies $ra<2h<4a$ and so $r \leqslant 3$. Similarly $|U_2|<|G|$ gives $l \leqslant 3$. We conclude: $2ah=4a(a-1)=|N|+|U_2|=ra+la^2 \leqslant 3a(a+1)$ giving $a \leqslant 7$ *i.e.* A is primary, contrary to lemma 1.

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