Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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A result on equiabsolute integrability

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **1** (1990), n.3, p. 189–194.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1990_9_1_3_189_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1990.

Rend. Mat. Acc. Lincei
s. 9, v. 1:189-194 (1990)

Calcolo delle variazioni. — *A result on equiabsolute integrability.* Nota di CRISTINA MARCELLI e ANNA SALVADORI, presentata (*) dal Socio L. CESARI.

ABSTRACT. — We prove the equiabsolute integrability of a class of gradients, for functions in $W^{1,1}$. The present result appears as the localized version of well-known classical theorems.

KEY WORDS: Equiabsolute integrability; Growth conditions; Calculus of variations.

RIASSUNTO. — *Un risultato di equiassoluta integrabilità.* Si prova un teorema di equiassoluta integrabilità per una classe di gradienti di funzioni in $W^{1,1}$, che si presenta come la versione localizzata di alcuni ben noti risultati classici.

Let Ω' be a class of gradients Dx of functions $x: G \to \mathbb{R}^n$, $G \subset \mathbb{R}^\nu$ open bounded, with $x \in W^{1,1}$, $n \geq 1$, $\nu \geq 1$.

As it is well known, the family Ω' is equiabsolutely integrable under suitable growth assumptions, (see Cesari [2, Theor. 10.4.i, ii, iii]). Here we take into consideration a «localization» of these growth conditions and prove that they still allow to obtain a «local» result of equiabsolute integrability. The present theorem finds interesting applications in problems of calculus of variations [3].

Let A be a given subset of the (t,x)-space $\mathbb{R}^{\nu+n}$ such that G is contained in the projection of A on the t-space $\mathbb{R}^\nu$; for every $t \in G$, we denote by $A(t) = \{x \in \mathbb{R}^n : (t,x) \in A\}$. For every $(t_0, x_0) \in A$ and $\sigma_0 > 0$, $t_0 = (t_0^1, \ldots, t_0^\nu)$ and $x_0 = (x_0^1, \ldots, x_0^n)$, let

$$U(t_0, \sigma_0) = \prod_{j=1}^{\nu} [t_0^j - \sigma_0, t_0^j + \sigma_0] \quad \text{and} \quad V(x_0, \sigma_0) = \prod_{i=1}^{n} [x_0^i - \sigma_0, x_0^i + \sigma_0].$$

We denote by $|E|$ the measure of a measurable subset E of $\mathbb{R}^\nu$.

Given a function $x \in W^{1,1}(G)$, for every $1 \leq i \leq n$ and $1 \leq j \leq \nu$, let $D^j x^i$ be the partial derivative of $x^i(t)$ with respect to t_j in the sense of distributions and let $Dx: G \to \mathbb{R}^{n\nu}$ be the gradient of x, *i.e.* $Dx = (D^j x^i, j = 1, \ldots, \nu, i = 1, \ldots, n) \in (L_1(G))^{n\nu}$.

Let $\tilde{\Omega}$ be a class of pairs of functions (η, x) with $\eta: G \to \mathbb{R}$, $\eta \in L_1(G)$ and $x: G \to \mathbb{R}^n$, $x \in W^{1,1}(G)$, such that $x(t) \in A(t)$ for every $t \in G$.

We denote by Ω_1 and Ω_2 the projections of $\tilde{\Omega}$ on the spaces $L_1(G)$ and $W^{1,1}(G)$ respectively, that is:

$$\Omega_1 = \{\eta \in L_1(G) : \exists x \in W^{1,1}(G) \text{ with } (\eta, x) \in \tilde{\Omega}\}$$

and

$$\Omega_2 = \{x \in W^{1,1}(G) : \exists \eta \in L_1(G) \text{ with } (\eta, x) \in \tilde{\Omega}\}.$$

Then we take $\Omega' = \{Dx : x \in \Omega_2\}$.

Let $(t_0, x_0) \in A$ be fixed with $t_0 \in G$. For every $\sigma_0 > 0$ and $x \in \Omega_2$ we set

$$E_{\sigma_0, x} = \{t \in U(t_0, \sigma_0) : x(t) \in V(x_0, \sigma_0)\}.$$

Let us consider now the following definition.

(*) Nella seduta del 10 marzo 1990.

DEFINITION 1. We shall say that the class Ω' is *equiabsolutely integrable at the point* (t_0, x_0) if there exists a constant $\rho_0 > 0$ such that the family

$$\{Dx|_{E_{\rho_0,x}} : Dx \in \Omega'\}$$

is equiabsolutely integrable; *i.e.* for every $\varepsilon > 0$ there exists a constant $\delta = \delta(t_0, x_0; \varepsilon) > 0$ such that given any set $F \subset G$ with $|F| < \delta$ we have

$$\int_{F \cap E_{\rho_0,x}} \|Dx(t)\| \, dt < \varepsilon .$$

We recall some growth conditions which are well known and frequently adopted in problems of calculus of variations and optimization theory. Actually we present here their localization.

DEFINITION 2. The class $\tilde{\Omega}$ is said to satisfy the *local growth condition* (g_1) *at the point* (t_0, x_0) if there are:

a) a constant $\rho_0 > 0$;

b) a Nagumo function $\varphi_0 : \mathbb{R}_0^+ \to \mathbb{R}$; *i.e.* $\varphi_0(\xi) \geq l_0$ for every $\xi \geq 0$ and

$$\lim_{\xi \to +\infty} \varphi_0(\xi)/\xi = +\infty ;$$

such that for every $(\eta, x) \in \tilde{\Omega}$ we have

$$\eta(t) \geq \varphi_0(\|Dx(t)\|) \tag{1}$$

for almost every $t \in E_{\rho_0,x}$.

DEFINITION 3. The class $\tilde{\Omega}$ is said to satisfy the *local growth condition* (g_2) *at the point* (t_0, x_0) if there is a constant $\rho_0 > 0$ such that: for every $\varepsilon > 0$ there is an L-integrable function $\psi_\varepsilon : U(t_0, \rho_0) \to \mathbb{R}_0^+$ such that for every $(\eta, x) \in \tilde{\Omega}$ we have

$$\|Dx(t)\| \leq \psi_\varepsilon(t) + \varepsilon\eta(t) \tag{2}$$

for almost every $t \in E_{\rho_0,x}$.

DEFINITION 4. The class $\tilde{\Omega}$ is said to satisfy the *local growth condition* (g_3) *at the point* (t_0, x_0) if there is a constant $\rho_0 > 0$ such that: for every vector $p \in \mathbb{R}^{n\nu}$ there is an L-integrable function $\phi_p : U(t_0, \rho_0) \to \mathbb{R}_0^+$ such that for every $(\eta, x) \in \tilde{\Omega}$ we have

$$\eta(t) \geq \langle p, Dx(t) \rangle - \phi_p(t) \tag{3}$$

for almost every $t \in E_{\rho_0,x}$.

Growth condition (g_1) is the localization of the classical Tonelli-Nagumo condition [6, 4]. Condition (g_2) has been introduced by Cesari in [1]; as it is well known, it is a weakening of condition (g_1) and it is equivalent to condition (g_3), which is due to Rockafellar [5] (see Cesari [2]).

Let us consider now the following growth condition which is inspired to those introduced by Tonelli in [7].

DEFINITION 5. We shall say that the class $\tilde{\Omega}$ satisfies the *local growth condition* (g_4) *at the point* (t_0, x_0) if there are:

a) three constants $\rho_0 > 0$, $\alpha_0 > 0$, $\mu_0 \geq 0$;

b) a continuous function $a_0: U_0 = U(t_0, \rho_0) \to \mathbb{R}_0^+$ with $a_0(t) > a_0(t_0) = 0$ for every $t \neq t_0$;

c) a monotone nondecreasing function $\psi_0: \mathbb{R}_0^+ \to \mathbb{R}_0^+$;

d) a function $\chi_0: [0, s_0] \to \mathbb{R}_0^+$, where $s_0 = \max_{t \in U_0} a_0(t)$, such that $\chi_0 \circ a_0 \in L_1$ and

$$\lim_{t \to t_0} a_0(t) \{\chi_0(a_0(t))\, \psi_0[\chi_0(a_0(t))]\}^{\alpha_0} = +\infty\,; \tag{4'}$$

such that for every $(\eta, x) \in \tilde{\Omega}$ we have

$$\eta(t) \geq a_0(t) \|Dx(t)\|^{1+\alpha_0} [\psi_0(\|Dx(t)\|)]^{\alpha_0} - \mu_0 \tag{4}$$

for almost every $t \in E_{\rho_0, x}$.

A comparison between condition (g_4) and the other ones is given in [3].

THEOREM 6. *Suppose that the class* $\tilde{\Omega}$ *satisfies at the point* (t_0, x_0) *anyone of the growth conditions* (g_i), $i = 1, \ldots, 4$. *Moreover assume that there exists* $M_0 > 0$ *such that, for every* $(\eta, x) \in \tilde{\Omega}$, *we have*

$$\int_{E_{\rho_0, x}} \eta(t)\, dt < M_0\,.$$

Then the class Ω' *is equiabsolutely integrable at the point* (t_0, x_0).

PROOF. Let us distinguish four cases.

a) First we suppose that growth condition (g_1) holds at (t_0, x_0). We consider the class $\overline{\Omega}$ of the functions $\overline{Dx}: U(t_0, \rho_0) \to \mathbb{R}^{n\nu}$ defined by

$$\overline{Dx}(t) = \begin{cases} Dx(t) & \text{for } t \in E_{\rho_0, x} \\ 0 & \text{for } t \in U(t_0, \rho_0) \setminus E_{\rho_0, x} \end{cases}$$

for every $x \in \Omega_2$.

Then for every $(\eta, x) \in \tilde{\Omega}$ from (1) we have

$$\int_{U(t_0, \rho_0)} \varphi_0(\|\overline{Dx}(t)\|)\, dt = \varphi_0(0) |U(t_0, \rho_0) \setminus E_{\rho_0, x}| + \int_{E_{\rho_0, x}} \eta(t)\, dt \leq (2\rho_0)^\nu |\varphi_0(0)| + M_0\,.$$

Thus, by virtue of equivalence theorem 10.3.i in Cesari [2], the class $\overline{\Omega}$ is equiabsolutely integrable and consequently the class Ω' is locally equiabsolutely integrable at the point (t_0, x_0).

b) Now we suppose that growth condition (g_2) holds at (t_0, x_0). Let $\psi_1: U_0 = U(t_0, \rho_0) \to \mathbb{R}_0^+$ be the L-integrable function given by (g_2) for $\varepsilon = 1$, and let

$$L_0 = M_0 + \int_{U_0} \psi_1(t)\, dt\,.$$

Let $\varepsilon > 0$ be fixed. We put $\sigma = \min\{1, \varepsilon/2L_0\}$ and consider the L-integrable function ψ_σ given by (g_2); then there is a constant $\delta = \delta(t_0, x_0; \sigma, \varepsilon) > 0$ such that for every $F \subset U_0$ with $|F| < \delta$ we have

$$\int_F \psi_\sigma(t)\,dt \leqslant \varepsilon/2\,.$$

Now let $F \subset U_0$ be any measurable set with $|F| < \delta$. Then for every $(\eta, x) \in \tilde{\Omega}$ we have that $\eta(t) + \psi_1(t) \geqslant 0$ on $E_{\rho_0,x}$ and therefore

$$\int_{F\cap E_{\rho_0,x}} \|Dx(t)\|\,dt \leqslant \int_{F\cap E_{\rho_0,x}} [\psi_\sigma(t) + \sigma\eta(t)]\,dt \leqslant$$
$$\leqslant \int_F \psi_\sigma(t)\,dt + \sigma \int_{E_{\rho_0,x}} [\eta(t) + \psi_1(t)]\,dt \leqslant \varepsilon/2 + \sigma L_0 = \varepsilon\,,$$

which proves the thesis.

c) Then we suppose that growth condition (g_3) holds at (t_0, x_0). Let $U_0 = U(t_0, \rho_0)$ and $\phi: U_0 \to \mathbb{R}_0^+$, $\psi: U_0 \to \mathbb{R}_0^+$ be the L-integrable functions of assumption (g_3) given in correspondence to the $n\nu$-vectors $u_1 = (1, 0, \ldots, 0)$ and $u_2 = (-1, 0, \ldots, 0)$. Then $D^1x^1(t) \leqslant \eta(t) + \phi(t)$ and $-D^1x^1(t) \leqslant \eta(t) + \psi(t)$, for *a.e.* $t \in E_{\rho_0,x}$, hence we have

1) $$0 \leqslant |D^1x^1(t)| \leqslant \eta(t) + \phi(t) + \psi(t), \qquad a.e. \text{ in } E_{\rho_0,x}\,.$$

Put

$$M_1 = \int_{U_0} [\phi(t) + \psi(t)]\,dt.$$

Let $\varepsilon > 0$ be fixed. Let $L > 0$ be an integer such that $n\nu M_0 L^{-1} \leqslant \varepsilon/3$ and $n\nu M_1 L^{-1} \leqslant \varepsilon/3$. If u_s, v_s denote the unit $n\nu$-vectors $u_s = (\delta_{sr}, r = 1, \ldots, n\nu)$, $v_s = (-\delta_{sr}, r = 1, \ldots, n\nu)$, then again by assumption (g_3), for $p = Lu_s$, and $p = Lv_s$, there are two L-integrable functions $\phi_s: U_0 \to \mathbb{R}_0^+$ and $\psi_s: U_0 \to \mathbb{R}_0^+$, such that $LD^jx^i(t) \leqslant \eta(t) + \phi_s(t)$, and $-LD^jx^i(t) \leqslant \eta(t) + \psi_s(t)$. Then for any $(\eta, x) \in \tilde{\Omega}$ we have

2) $$0 \leqslant L|(Dx)_s| \leqslant \eta(t) + \phi_s(t) + \psi_s(t) \qquad a.e. \text{ in } E_{\rho_0,x}\,,$$

$s = 1, \ldots, n\nu$, where $(Dx)_s$ is the s-th component of the vector Dx. Let $\Phi_0: U_0 \to \mathbb{R}_0^+$ and $\Psi_0: U_0 \to \mathbb{R}_0^+$ be the L-integrable functions defined by

$$\Phi_0(t) = \sum_{s=1}^{\nu n} \phi_s(t) \quad \text{and} \quad \Psi_0(t) = \sum_{s=1}^{\nu n} \psi_s(t)\,;$$

then from 2) we have

3) $$L\|Dx(t)\| \leqslant n\nu\,\eta(t) + \Phi_0(t) + \Psi_0(t)\,, \qquad a.e. \text{ in } E_{\rho_0,x}\,.$$

Moreover there is a constant $\delta = \delta(t_0, x_0, \rho_0; \varepsilon) > 0$ such that if F is a subset of U_0 with $|F| < \delta$ then

4) $$\int_F [\Phi_0(t) + \Psi_0(t)]\,dt < \varepsilon/3\,.$$

Then from 3), 1) and 4), we have that for every $Dx \in \Omega'$

$$\int_{F\cap E_{\rho_0,x}} \|Dx(t)\|\,dt \leqslant L^{-1} n\nu \int_{F\cap E_{\rho_0,x}} \eta(t)\,dt + L^{-1} \int_F [\Phi_0(t) + \Psi_0(t)]\,dt \leqslant$$

$$\leq L^{-1} n\nu \int_{E_{\rho_0,x}} [\eta(t) + \phi(t) + \psi(t)]\, dt + L^{-1} \int_F [\Phi_0(t) + \Psi_0(t)]\, dt \leq$$

$$\leq L^{-1} n\nu M_0 + L^{-1} n\nu M_1 + L^{-1} \varepsilon/3 \leq \varepsilon ,$$

which proves the assertion.

d) Finally we suppose that growth condition (g_4) holds at (t_0, x_0). Let $\varepsilon > 0$ be fixed. From the hypothesis $\chi_0 \circ a_0 \in L_1$ it follows that there is a constant $\delta_1 = \delta_1(\varepsilon) > 0$ such that for every $F \subset U_0$ with $|F| < \delta_1$, we have

5) $$\int_F \chi_0(a_0(t))\, dt < \varepsilon/3 .$$

Moreover from (4′) there is a constant $0 < r = r(\varepsilon) < \rho_0$ such that if $\|t - t_0\| < r$, we have

6) $$a_0(t)\{\chi_0(a_0(t))\, \psi_0[\chi_0(a_0(t))]\}^{\alpha_0} > 3(M_0 + 2\rho_0\mu_0)/\varepsilon .$$

Now, by the monotonicity of ψ_0 it follows that

$$\lim_{y \to +\infty} y\psi_0(y) = +\infty ;$$

then put $m = \min\{a_0(t), t \in U_0 \setminus U(t_0, r)\} > 0$, there is a constant $0 < \bar{y} = \bar{y}(\varepsilon, r) = \bar{y}(\varepsilon)$ such that

7) $$[y\psi_0(y)]^{\alpha_0} > 3(M_0 + 2\rho_0\mu_0)/(m\varepsilon) \qquad \text{for every } y > \bar{y} .$$

Let $\delta = \delta(\varepsilon) = \min\{\delta_1, \varepsilon/3\bar{y}\}$ and let $F \subset E_{\rho_0,x}$ be fixed with $|F| < \delta$. For any function $Dx \in \Omega'$ we set

$$F_1 = \{t \in F : \|Dx(t)\| \leq \bar{y}\}; \qquad F_2 = \{t \in F : \|Dx(t)\| \leq \chi_0(a_0(t))\};$$

$$F_3 = [F \setminus (F_1 \cup F_2)] \cap U(t_0, r); \qquad F_4 = F \setminus (F_1 \cup F_2 \cup F_3).$$

From (4) and the monotonicity of ψ_0, it follows that

$$\eta(t) \geq a_0(t)\|Dx(t)\|\{\chi_0(a_0(t))\, \psi_0[\chi_0(a_0(t))]\}^{\alpha_0} - \mu_0$$

for *a.e.* $t \in F \setminus F_2$; and then, by virtue of 6), we have

8) $$\|Dx(t)\| < [\eta(t) + \mu_0]\, \varepsilon/3(M_0 + 2\rho_0\mu_0) \qquad \text{for } a.e.\ t \in F_3 .$$

Again from (4) it follows that $\eta(t) \geq m\|Dx(t)\|\, [\|Dx(t)\|\, \psi_0(\|Dx(t)\|)]^{\alpha_0} - \mu_0$ for *a.e.* $t \in F \setminus U(t_0, r)$ and then, taking into account of 7), we have

9) $$\|Dx(t)\| < m\varepsilon(\eta(t) + \mu_0)/3m(M_0 + 2\rho_0\mu_0)$$

for *a.e.* $t \in F \setminus (F_1 \cup U(t_0, r))$.

Finally, by 8), 9) and 5), we have

$$\int_F \|Dx(t)\|\, dt \leq \bar{y}|F_1| + \int_{F_2} \chi_0(a_0(t))\, dt + [\varepsilon/3(M_0 + 2\rho_0\mu_0)] \int_{F_3 \cup F_4} [\eta(t) + \mu_0]\, dt \leq$$

$$\leq \bar{y}\, \varepsilon/3\bar{y} + \varepsilon/3 + \varepsilon/3 = \varepsilon ,$$

which concludes the proof.

The following result is a slightly modified version of Theorem 6.

THEOREM 6′. *Suppose that the class $\tilde{\Omega}$ satisfies at the point (t_0, x_0) anyone of the growth conditions (g_i), $i = 1, \ldots, 4$. Moreover assume that the functions $\eta \in \Omega_1$ are equibounded in $L_1[U(t_0, \rho_0)]$.*

Then the class Ω' is equiabsolutely integrable at the point (t_0, x_0).

COROLLARY 7. *Let A be compact and suppose that the class $\tilde{\Omega}$ has the property that at every point $(t_0, x_0) \in A$ one of the growth conditions (g_i), $i = 1, \ldots, 4$, holds (not necessarily the same). Moreover suppose that the class Ω_1 is equibounded in $L_1(G)$.*

Then the class Ω' is equiabsolutely integrable in G.

REMARK 8. Note that the assumption that Ω_1 is equibounded in $L_1(G)$ is satisfied if we know that there exist a constant $L > 0$ and a function $h \in L_1(G)$ such that for every $\eta \in \Omega_1$: $\eta(t) \geqslant h(t)$ for almost every $t \in G$ and $\int_G \eta(t)\,dt \leqslant L$.

Indeed we have

$$\int_G |\eta(t)|\,dt = \int_G \eta(t)\,dt + 2\int_G \eta^-(t)\,dt \leqslant L + 2\int_G |h(t)|\,dt = M\,.$$

In [3] we shall present a problem of calculus of variations where different local growth conditions are assumed and for which our results imply the existence of the absolute minimum.

This research has been carried out within the GNAFA-CNR.

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