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Holomorphic automorphisms of the unit ball of a direct sum

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Geometria. — *Holomorphic automorphisms of the unit ball of a direct sum.* Nota di CARLO PETRONIO, presentata (*) dal Socio E. VESENTINI.

ABSTRACT. — We endow the direct sum of two complex Banach spaces with a suitable norm, and we investigate the orbit of the origin for the group of holomorphic automorphisms of the outcoming unit ball.

KEY WORDS: Direct sum; p-norm; Orbit of the origin.

RIASSUNTO. — *Automorfismi olomorfi della palla unitaria di una somma diretta.* La somma diretta di due spazi di Banach complessi viene dotata di una opportuna norma, e viene studiata l'orbita dell'origine rispetto al gruppo degli automorfismi olomorfi della palla unitaria risultante.

In 1928 Kritikos [4] proved that every holomorphic automorphism of the domain $\{z: |z_1| + |z_2| < 1\} \subset \boldsymbol{C}^2$ fixes the origin. This result was re-obtained in 1931 by Thullen as a consequence of his investigations on a more general class of domains in $\boldsymbol{C}^2$ (see [7]). In recent years Vesentini [8, 9] and Braun, Kaup and Upmeier [1] proved that the same result holds for the unit ball of an L^p space, provided $p \neq 2, \infty$ and the space is at least two-dimensional.

In this paper we consider a situation which generalizes in a natural way the one envisaged by Thullen: given two complex Banach spaces F and G we study the orbit of the origin for the group of holomorphic automorphisms of a domain $D \subset F \times G$ which is the unit ball for a suitable continuous norm.

In section 1 we obtain a general result which can be translated in euristic terms as «the orbit of the origin in the unit ball of a direct sum cannot exceed the product of the orbits in the two addenda». In sections 2 and 3 we consider the special case of p-norms (for $1 \leqslant p \leqslant \infty$) and we determine the orbit of the origin in almost every case; in particular we obtain an analogue of the theorem proved by Vesentini and Braun, Kaup and Upmeier. In section 4 we apply the results of 2 and 3 to the concrete case of direct sums of L^p-spaces. A somewhat similar situation was considered in [5], where Thullen's results were generalized to certain compact operator spaces.

1. In the following F and G will be positive-dimensional Banach spaces over $\boldsymbol{C}$. Symbols as f and g will denote elements of F and G respectively. If $1 \leq p \leq \infty$ we will denote by $F \oplus_p G$ the direct sum of F and G endowed with the norm $\|\cdot\|_p$ (which we will call «p-norm» or «norm of order p»), given by $\|(f, g)\|_p = (\|f\|^p + \|g\|^p)^{1/p}$ for $1 \leq p < \infty$, and by $\|(f, g)\|_p = \max\{\|f\|, \|g\|\}$ when $p = \infty$.

More in general, if $\|\cdot\|$ is a norm defining the product topology on $F \times G$, we will denote by $F \oplus_{\|\cdot\|} G$ the direct sum of F and G endowed with this norm.

The essential tool for our results is a theorem proved by Stachó in [6] as a corollary of the general theory of bounded circular domains, first developed by Kaup and

(*) Nella seduta del 10 febbraio 1990.

Upmeier in [3]. We recall that, for a Banach space F, $\mathcal{L}_s^2(F)$ denotes the space of continuous bi-linear symmetric functions from $F \times F$ to F; for $Q \in \mathcal{L}_s^2(F)$ the mapping $f \mapsto Q(f, f)$ defines a generic continuous homogeneous polynomial of degree two on F (and Q is uniquely determined by this polynomial). Hence the elements of $\mathcal{L}_s^2(F)$ will be often referred to as polynomials.

Definition: if D is a domain in F, the set of all completely integrable (or complete) holomorphic vector fields $X: D \to F$ will be denoted by $aut(D)$.

Theorem 0: if F is a complex Banach space, B is its open unit ball and $Aut(B)$ denotes the group of all holomorphic automorphisms of B, then $Aut(B)(0) = F_0 \cap B$, where F_0 is a closed $\mathbf{C}$-linear subspace of F. Moreover, given $c \in F$, we have $c \in F_0$ if, and only if, $\exists\, Q_c \in \mathcal{L}_s^2(F)$ such that one of the following equivalent conditions is fulfilled:

(1) $(f \mapsto c - Q_c(f, f)) \in aut(B)$;

(2) if $a \in F$ and $\phi \in F^*$ are such that $\phi(a) = \|a\| \cdot \|\phi\|$, then $\phi(Q_c(a, a)) = \|a\|^2 \cdot \overline{\phi(c)}$.

If $c \in F_0$ the polynomial Q_c fulfilling these conditions is unique, and the mapping $c \mapsto Q_c$ from F_0 to $\mathcal{L}_s^2(F)$ is continuous and anti-linear.

It will sometimes be useful to observe that property (2) in theorem 0 is equivalent to (3) if $a \in F$ and $\phi \in F^*$ are such that $\phi(a) = \|a\| = \|\phi\| = 1$, then $\phi(Q_c(a, a)) = \overline{\phi(c)}$.

Henceforth, given a complex Banach space F, we will always denote by F_0 the subspace of F described in the above theorem. Moreover, for $c \in F_0$, we will often refer to Q_c as the «polynomial relative to c».

All our results will deal with the subspace F_0 and not with the orbit of the origin itself, which is linked to F_0 by theorem 0.

Before proving our first general result we record the following elementary:

Lemma 1: if $1 \le p \le \infty$ and q is the conjugate exponent of p, then $(F \oplus_p G)^*$ is isometrically isomorphic to $F^* \oplus_q G^*$.

Theorem 1: if the norm $\|(f, g)\|$ of any pair (f, g) depends only on $\|f\|_F$ and $\|g\|_G$, then $(F \oplus_{\|\cdot\|} G)_0 \subseteq F_0 \times G_0$.

Proof: the proof will be carried out in three steps.

Step 1: there exist two positive numbers α and β such that

$$\begin{cases} \|(f, 0)\| = \alpha \|f\| \\ \|(0, g)\| = \beta \|g\| \\ \|(f, g)\| \ge \max(\alpha \|f\|, \beta \|g\|) \end{cases} \qquad \forall (f, g) \in F \times G.$$

Since $f \mapsto \|(f, 0)\|$ is a norm on F and it has the form $f \mapsto l(\|f\|)$, with $l: \mathbf{R}_+ \to \mathbf{R}_+$, chosen $f_0 \in F$ with $\|f_0\| = 1$ and set $\alpha = l(1)$ we have for $\lambda \ge 0$

$$l(\lambda) = l(\|\lambda f_0\|) = \|(\lambda f_0, 0)\| = \lambda \|(f_0, 0)\| = \lambda l(\|f_0\|) = \alpha\lambda.$$

It follows that $\|(f,0)\| = \alpha\|f\|$ and similarly $\|(0,g)\| = \beta\|g\|$. If $(f,g) \in F \times G$ we have

$$\alpha\|f\| = \|(f,0)\| = \|(f,g)/2 + (f,-g)/2\| \leq \|(f,g)\|/2 + \|(f,-g)\|/2 = \|(f,g)\|.$$

Similarly $\beta\|g\| \leq \|(f,g)\|$ and step 1 is established.

Step 2: we can assume $\alpha = \beta = 1$.

Let $\widetilde{F}$ $(\widetilde{G})$ denote the Banach space which coincides with F (G) as a linear space and whose norm is defined by $\|f\|_{\widetilde{F}} = \alpha\|f\|_F$ $(\|g\|_{\widetilde{G}} = \beta\|g\|_G)$. It is easily verified that $\widetilde{F}_0 = F_0$ and $\widetilde{G}_0 = G_0$.

Suppose the theorem is true when $\alpha = \beta = 1$; then $(F \oplus_{\|\cdot\|} G)_0 \subseteq \widetilde{F}_0 \times \widetilde{G}_0 = F_0 \times G_0$ and hence the theorem is true in any case.

Step 3: conclusion.

Suppose $(c_1, c_2) \in (F \oplus_{\|\cdot\|} G)_0$ and let $Q = Q_{(c_1, c_2)} \in \mathcal{L}_s^2(F \oplus_{\|\cdot\|} G)$ be the polynomial relative to it. By symmetry, it is enough to verify that $c_1 \in F_0$. Let π_1 be the natural projection of $F \oplus_{\|\cdot\|} G$ onto F and set $Q_1(f_1, f_2) = (\pi_1 \circ Q)((f_1, 0), (f_2, 0))$; Q_1 belongs to $\mathcal{L}_s^2(F)$. Now, if $f_0 \in F$ and $f_0^* \in F^*$ are such that $f_0^*(f_0) = \|f_0\| \cdot \|f_0^*\|$, we set $a = (f_0, 0)$ and $\phi = (f_0^*, 0)$. Obviously $a \in F \oplus_{\|\cdot\|} G$ and $\phi \in (F \oplus_{\|\cdot\|} G)^*$; moreover it follows from the properties of $\|\cdot\|$ that $\|a\| = \|f_0\|$ and

$$\|\phi\| = \sup_{(f,g)\neq 0} \frac{|f_0^*(f)|}{\|(f,g)\|} \geq \sup_{f\neq 0} \frac{|f_0^*(f)|}{\|(f,0)\|} = \sup_{f\neq 0} \frac{|f_0^*(f)|}{\|f\|} = \|f_0^*\|,$$

$$\|\phi\| = \sup_{(f,g)\neq 0} \frac{|f_0^*(f)|}{\|(f,g)\|} \leq \sup_{(f,g)\neq 0} \frac{|f_0^*(f)|}{\max(\|f\|, \|g\|)} \leq \sup_{f\neq 0} \frac{|f_0^*(f)|}{\|f\|} = \|f_0^*\|;$$

hence $\|\phi\| = \|f_0^*\|$; moreover $\phi(a) = f_0^*(f_0)$ and therefore $\phi(a) = \|a\| \cdot \|\phi\|$.

Thus the hypothesis about (c_1, c_2) and Q applies to the pair a, ϕ:

$$(f_0^*, 0)\, Q((f_0, 0), (f_0, 0)) = \|(f_0, 0)\|^2 \cdot \overline{(f_0^*, 0)(c_1, c_2)}.$$

Recalling the definition of Q_1, this formula implies that

$$f_0^*(Q_1(f_0, f_0)) = \|f_0\|^2 \cdot \overline{f_0^*(c_1)}.$$

Since this identity holds for every pair $f_0 \in F$, $f_0^* \in F^*$ with the only condition that $f_0^*(f_0) = \|f_0\| \cdot \|f_0^*\|$, it follows that $c_1 \in F_0$ and the proof is complete. □

We remark that the hypothesis in theorem 1 is quite natural: it means that we consider on the direct sum the norms «obtained from the original norms». Without any hypothesis on the norm this theorem is certainly false (for a counter-example, consider the finite-dimensional case); however our hypothesis can be slightly weakened (as it is evident from the proof).

2. We now turn to the p-norms on the direct sum. Our first result generalizes Kritikos' theorem and it can be considered as an analogue of the results proved in [1], [8] and [9].

THEOREM 2: if $1 \leq p < \infty$ and $p \neq 2$, then $(F \oplus_p G)_0 = \{0\}$.

PROOF: let $E=F\oplus_p G$; suppose $(f,g)\in E_0$ and let $Q=Q_{(f,g)}\in\mathcal{L}_s^2(E)$ be the polynomial relative to it. Since F and G can be interchanged it is enough to prove that $f=0$. Choose $g_1\in G\setminus\{0\}$. By the Hahn-Banach theorem we can find $f^*\in F^*$ and $g^*\in G^*$ such that $\|f^*\|=\|g^*\|=1$, $f^*(f)=\|f\|$ and $g^*(g_1)=\|g_1\|$.

For arbitrary $\rho>0$ and $\theta\in\mathbf{R}$ we define

$$a=(f,\rho e^{i\theta}g_1),\qquad \phi=(\|f\|^{p-1}f^*,\rho^{p-1}\|g_1\|^{p-1}\cdot e^{-i\theta}g^*),$$

(with the usual convention $0^0=1$). We claim that $\phi(a)=\|a\|\cdot\|\phi\|$; in fact for $p=1$,

$$\|a\|_1=\|f\|+\rho\|g_1\|,\quad \|\phi\|_\infty=1,\quad \phi(a)=f^*(f)+g^*(\rho g_1)=\|f\|+\rho\|g_1\|=\|a\|_1\cdot\|\phi\|_\infty;$$

if $p>1$ (and q is the conjugate exponent of p),

$$\|a\|_p=(\|f\|^p+\rho^p\cdot\|g_1\|^p)^{1/p},$$

$$\|\phi\|_q=(\|f\|^{q(p-1)}\cdot\|f^*\|^q+\rho^{q(p-1)}\cdot\|g_1\|^{q(p-1)}\cdot\|g^*\|^q)^{1/q}=(\|f\|^p+\rho^p\cdot\|g_1\|^p)^{1-1/p},$$

$$\phi(a)=\|f\|^p+\rho^p\cdot\|g_1\|^p=\|a\|_p\cdot\|\phi\|_q.$$

Our claim is proved; hence the hypothesis on (f,g) and Q applies to the pair a, ϕ:

$$(\|f\|^{p-1}f^*,\rho^{p-1}\|g_1\|^{p-1}\cdot e^{-i\theta}g^*)\,Q((f,\rho e^{i\theta}g_1),(f,\rho e^{i\theta}g_1))=$$

$$=(\|f\|^p+\rho^p\|g_1\|^p)^{2/p}\,\overline{(\|f\|^{p-1}f^*,\rho^{p-1}\|g_1\|^{p-1}\cdot e^{-i\theta}g^*)(f,g)}\,,$$

i.e. by direct computation $\alpha_2e^{2i\theta}+\alpha_1e^{i\theta}+\alpha_0+\alpha_{-1}e^{-i\theta}=0\ \forall\theta\in\mathbf{R}$, where the numbers α_j are independent of θ, and therefore they are all zero.

A straight-forward calculation proves that if we set

$$\lambda=\|f\|^{p-1}(f^*,0)(Q((f,0),(f,0))),\qquad \mu=2\|g_1\|^{p-1}(0,g^*)(Q((f,0),(0,g_1))),$$

the identity $\alpha_0=0$ can be re-written as

$$\lambda=\mu\cdot\rho^p-(\|f\|^p+\rho^p\|g_1\|^p)^{2/p}\|f\|^p=0\qquad\forall\rho>0.$$

We can now divide by ρ^2 and pass to the limit as $\rho\to\infty$; we obtain that

$$\lim_{\rho\to\infty}\mu\cdot\rho^{p-2}$$

exists and equals $\|g_1\|^2\cdot\|f\|^p$. Since $p\neq 2$ this limit must vanish; but $g_1\neq 0$, then $f=0$ and the proof is complete. □

The case $p=\infty$ is not new, and will be included for the sake of completeness.

THEOREM 3: $(F\oplus_\infty G)_0=F_0\times G_0$.

PROOF: inclusion $\subseteq$ follows at once from theorem 2. For the opposite inclusion, it is not difficult to verify that if $f\in F_0$ ($g\in G_0$) and Q_f (Q_g) is the polynomial relative to it, then $(f,g)\in(F\oplus_\infty G)_0$ and $Q_{(f,g)}(f_1,g_1)=(Q_f(f_1),Q_g(g_1))$. □

3. For $p=2$ the determination of the orbit of the origin is much more complicated (it will follow from our results that no general theorem like 2 or 3 can hold in this case).

We will examine some relevant particular cases; the first one is when one of the spaces involved is a Hilbert space. We begin by proving the well-known result of homogeneity of the unit ball of a Hilbert space and we determine the mapping $c \mapsto Q_c$ of theorem 0.

LEMMA 2: if H is a Hilbert space then $H_0 = H$ and for $c \in H$ we have $Q_c(x, y) = ((x|c)\, y + (y|c)\, x)/2$.

PROOF: we recall that $H^* \cong H$ (anti-linearly). Moreover

$$(a|\phi) = \|a\| = \|\phi\| = 1 \Leftrightarrow \phi = a, \qquad \|a\| = 1 .$$

Since the map Q_c defined above belongs to $\mathcal{L}^2_s(H)$ it is enough to show that if $\|a\| = 1$ then $((a|c)\, a|a) = \overline{(c|a)}$, and this is obvious. □

We can now prove the first significant result in the case $p = 2$:

THEOREM 4: if H is a Hilbert space then $(H \oplus_2 F)_0 \supseteq H \times \{0\}$; moreover if $x \in H$ we have $Q_{(x,0)}((y_1, f_1), (y_2, f_2)) = ((y_1|x) \cdot (y_2, f_2) + (y_2|x) \cdot (y_1, f_1))/2$.

PROOF: given $x \in H$, we must prove that $Q_{(x,0)}$ exists (and is given by the above formula). Therefore, let $a = (y, f)$ and $\phi = ((\cdot|z), f^*))$ be such that $\|a\| = \|\phi\| = \phi(a) = 1$; it is easily verified that $z = y$, hence we only have to prove that $((\cdot|y), f^*)((y|x) \cdot (y, f)) = \overline{((\cdot|y), f^*)(x, 0)}$.

This is quite obvious since the first member can be developed as

$$(y|x) \cdot ((y|y) + f^*(f)) = (y|x)\, \phi(a) = (y|x) = \overline{(x|y)} = \overline{((\cdot|y), f^*)(x, 0)} . \qquad \square$$

We consider now the case when one of the spaces involved in a direct sum with 2-norm is a commutative C^*-algebra with identity. As we will see, the behaviour of these spaces is opposite to that of Hilbert spaces. This is quite surprising since (as widely known) the unit ball of a commutative C^*-algebra with identity shares with that of a Hilbert space the property of being homogeneous.

We determine now the function $c \mapsto Q_c$ of theorem 0.

LEMMA 3: if $\mathcal{A}$ is a commutative C^*-algebra with identity then $\mathcal{A}_0 = \mathcal{A}$ and for $c \in \mathcal{A}$ $Q_c(a, b) = c^* \cdot a \cdot b$.

PROOF: by the Gel'fand-Naimark representation theorem (see *e.g.* [2, pp. 18-28]), we can suppose $\mathcal{A} = C(\mathcal{T})$ where $\mathcal{T}$ is compact and Hausdorff. Let B be the open unit ball of $C(\mathcal{T})$. Given $c \in C(\mathcal{T})$ and $a_0 \in B$, we have to prove that the Cauchy problem

$$\begin{cases} a'(t) = c - \bar{c} \cdot a(t)^2 \\ a(0) = a_0 \end{cases}$$

has a solution $a: \mathbf{R} \to B$. If we define $s: \mathcal{T} \to \mathbf{C}$ by

$$s(\tau) = \begin{cases} c(\tau)/|c(\tau)| & \text{if } c(\tau) \neq 0 \\ 1 & \text{otherwise}, \end{cases}$$

the solution is explicitly given by $a(t) = (s \cdot \tanh(t|c|) + a_0)/(1 + \bar{s} \cdot \tanh(t|c|) \cdot a_0)$. □

LEMMA 4: if $E = \mathbf{C} \oplus_2 (\mathbf{C} \oplus_\infty \mathbf{C})$, then $E_0 = \mathbf{C} \times \{0\}$.

PROOF: by theorem 4, $E_0 \supseteq \mathbf{C} \times \{0\}$ hence we only have to prove that $E_0 \cap (\{0\} \times \mathbf{C}^2) = \{0\}$. Suppose $c = {}^t(0, c_2, c_3) \in E_0$ and let $Q = {}^t(Q_1, Q_2, Q_3)$ be the polynomial relative to it (for brevity, we will write $Q_i(z)$ instead of $Q_i(z, z)$).

It is easily verified that the following isometric isomorphisms hold:

$$E \cap (\mathbf{C}^2 \times \{0\}) \cong \mathbf{C} \oplus_2 \mathbf{C}, \tag{1}$$

$$E \cap (\mathbf{C} \times \{0\} \times \mathbf{C}) \cong \mathbf{C} \oplus_2 \mathbf{C}, \tag{2}$$

$$E \cap (\{0\} \times \mathbf{C}^2) \cong \mathbf{C} \oplus_\infty \mathbf{C}. \tag{3}$$

It follows respectively that:

(1)′ $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} Q_1(z_1, z_2, 0) \\ Q_2(z_1, z_2, 0) \end{pmatrix}$ is the polynomial relative to ${}^t(0, c_2)$ in $\mathbf{C} \oplus_2 \mathbf{C}$;

(2)′ $\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} Q_1(z_1, 0, z_3) \\ Q_3(z_1, 0, z_3) \end{pmatrix}$ is the polynomial relative to ${}^t(0, c_3)$ in $\mathbf{C} \oplus_2 \mathbf{C}$;

(3)′ $\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} Q_2(0, z_2, z_3) \\ Q_3(0, z_2, z_3) \end{pmatrix}$ is the polynomial relative to ${}^t(c_2, c_3)$ in $\mathbf{C} \oplus_\infty \mathbf{C}$.

Using lemmas 2 and 3, these three relations imply respectively:

$$\tag{1$''$} \begin{cases} Q_1(z_1, z_2, 0) = \overline{c_2}\, z_1 z_2 \\ Q_2(z_1, z_2, 0) = \overline{c_2}\, z_2^2, \end{cases}$$

$$\tag{2$''$} \begin{cases} Q_1(z_1, 0, z_3) = \overline{c_3}\, z_1 z_3 \\ Q_3(z_1, 0, z_3) = \overline{c_3}\, z_3^2, \end{cases}$$

$$\tag{3$''$} \begin{cases} Q_2(0, z_2, z_3) = \overline{c_2}\, z_2^2 \\ Q_3(0, z_2, z_3) = \overline{c_3}\, z_3^2. \end{cases}$$

From these formulas we deduce that there exist complex numbers α, β and γ such that

$$\begin{cases} Q_1(z_1, z_2, z_3) = \alpha z_2 z_3 + \overline{c_2}\, z_1 z_2 + \overline{c_3}\, z_1 z_3 \\ Q_2(z_1, z_2, z_3) = \beta z_1 z_3 + \overline{c_2}\, z_2^2 \\ Q_3(z_1, z_2, z_3) = \gamma z_1 z_2 + \overline{c_3}\, z_3^2. \end{cases}$$

Setting, for $\rho > 0$,

$$a = \begin{bmatrix} \rho \\ 1 \\ 1 \end{bmatrix} \in E, \qquad \phi = \begin{bmatrix} \rho \\ 1 \\ 0 \end{bmatrix} \in E^*,$$

it is easily checked that $\|a\| = \|\phi\| = (1+\rho^2)^{1/2}$ and $\phi(a) = 1+\rho^2 = \|a\|\cdot\|\phi\|$, hence the formula which characterizes Q applies to the pair a, ϕ: $\phi(Q(a)) = \|a\|^2\cdot\overline{\phi(c)}$, whence, by the above expression of Q, we obtain:

$$\rho Q_1(\rho,1,1) + Q_2(\rho,1,1) = (1+\rho^2)\,\overline{c_2} \Rightarrow \rho\alpha + \rho^2\overline{c_2} + \rho^2\overline{c_3} + \beta\rho + \overline{c_2} = \overline{c_2} + \rho^2\overline{c_2} \Rightarrow$$
$$\Rightarrow \rho^2\overline{c_3} + \rho(\alpha+\beta) = 0\,.$$

Since $\rho > 0$ is arbitrary, we must have $c_3 = 0$.

Similarly $c_2 = 0$ and the proof is complete. □

Theorem 5: if $\mathcal{A}$ is a commutative C^*-algebra with identity and $\dim_C(\mathcal{A}) \geq 2$, then $(F \oplus_2 \mathcal{A})_0 \subseteq F_0 \times \{0\}$.

Proof: by the theorem 1 it suffices to verify that $(F\oplus_2\mathcal{A})_0 \subseteq F\times\{0\}$. Without any loss in generality, we can suppose, as before, $\mathcal{A} = C(\mathcal{T})$ where $\mathcal{T}$ is a Hausdorff compact topological space. We set $G = F\oplus_2 C(\mathcal{T})$.

Let $(f_0, c) \in G_0$ and let $Q = Q_{(f_0,c)}$ be the homogeneous polynomial of degree two relative to it. Given $\tau_2 \in \mathcal{T}$ we want to prove that $c(\tau_2) = 0$.

Since $\dim C(\mathcal{T}) \geq 2$ there exists $\tau_3 \in \mathcal{T}\setminus\{\tau_2\}$; moreover, since $\mathcal{T}$ is a Hausdorff space, we can find disjoint open neighborhoods U_2 and U_3 of τ_2 and τ_3 respectively. Urysohn's lemma implies that there exist continuous functions $a_2, a_3 : \mathcal{T} \to [0,1]$ such that $a_i^{-1}(1) = \{\tau_i\}$, $a_i^{-1}(0) = \mathcal{T}\setminus U_i$ for $i = 2, 3$.

We remark that by construction $a_2 a_3 = 0$.

Finally, we choose $f_1 \in F$ and $f_1^* \in F^*$ such that $\|f_1\| = \|f_1^*\| = f_1^*(f_1) = 1$. Let $E = \mathbf{C}\oplus_2(\mathbf{C}\oplus_\infty \mathbf{C})$ be the space considered in the above lemma. If $z\in E$ and $\phi \in E^*$ are such that $\|z\| = \|\phi\| = \phi(z) = 1$ we set

$$\tilde{z} = (z_1 f_1, z_2 a_2 + z_3 a_3) \in G, \qquad \tilde{\phi}(f,a) = \phi_1 f_1^*(f) + \phi_2 a(\tau_2) + \phi_3 a(\tau_3)\,, \qquad \tilde{\phi}\in G^*.$$

It is easily checked that $\|\tilde{z}\|^2 = |z_1|^2 + (\max\{|z_2|,|z_3|\})^2 = \|z\|^2 = 1$,

$$\|\tilde{\phi}\|^2 = |\phi_1|^2 + (|\phi_2| + |\phi_3|)^2 = \|\phi\|^2 = 1, \qquad \tilde{\phi}(\tilde{z}) = \phi_1 z_1 + \phi_2 z_2 + \phi_3 z_3 = \phi(z) = 1.$$

It follows that the hypothesis about (f_0, c) and Q applies to the pair $\tilde{z}, \tilde{\phi}$: if we set $Q = (Q_F, Q_{\mathcal{A}})$ we have

$$(*)\qquad \phi_1 f_1^*(Q_F(z_1 f_1, z_2 a_2 + z_3 a_3)) + \phi_2\, Q_{\mathcal{A}}(z_1 f_1, z_2 a_2 + z_3 a_3)(\tau_2) +$$
$$+ \phi_3\, Q_{\mathcal{A}}(z_1 f_1, z_2 a_2 + z_3 a_3)(\tau_3) = \overline{\phi_1 f_1^*(f_0) + \phi_2 c(\tau_2) + \phi_3 c(\tau_3)}\,.$$

The function P defined by

$$P\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} f_1^* Q_F(w_1 f_1, w_2 a_2 + w_3 a_3) \\ Q_{\mathcal{A}}(w_1 f_1, w_2 a_2 + w_3 a_3)(\tau_2) \\ Q_{\mathcal{A}}(w_1 f_1, w_2 a_2 + w_3 a_3)(\tau_3) \end{bmatrix},$$

is a continuous homogeneous polynomial of degree 2 on E whose definition is independent of z and ϕ, for which $(*)$ can be re-written as

$$\phi(P(z)) = \overline{\phi\begin{bmatrix} f_1^*(f_0) \\ c(\tau_2) \\ c(\tau_3) \end{bmatrix}}.$$

This identity holds for all pairs $z \in E$, $\phi \in E^*$ subject only to the condition $\|z\| = \|\phi\| = \phi(z) = 1$. Hence

$$\begin{bmatrix} f_1^*(f_0) \\ c(\tau_2) \\ c(\tau_3) \end{bmatrix} \in E_0 \Rightarrow c(\tau_2) = 0. \quad \square$$

The same technique used for theorem 5 leads to the proof of the following:

THEOREM 6: if F_1, F_2 and F_3 are positive-dimensional Banach spaces then $(F_1 \oplus_2 (F_2 \oplus_\infty F_3))_0 \subseteq F_1 \times \{0\}$.

PROOF: let $G = F_1 \oplus_2 (F_2 \oplus_\infty F_3)$. For $c = (c_1, c_2, c_3) \in G_0$ let $Q = Q_c$ be the homogeneous polynomial of degree two relative to c. For $i = 1,2,3$ we set

$$\hat{c}_i = \begin{cases} c_i/\|c_i\| & \text{if } c_i \neq 0, \\ \text{an arbitrary unit vector of } F_i & \text{if } c_i = 0. \end{cases}$$

We choose now for $i = 1,2,3$ linear functionals $k_i \in F_i^*$ with $k_i(\hat{c}_i) = \|k_i\| = 1$. Let E be the space of lemma 4; if $z \in E$, $\phi \in E^*$ are such that $\|z\| = \|\phi\| = \phi(z) = 1$, we set $\tilde{z} = (z_1 \hat{c}_1, z_2 \hat{c}_2, z_3 \hat{c}_3) \in G$, $\tilde{\phi} = (\phi_1 k_1, \phi_2 k_2, \phi_3 k_3) \in G^*$.

It is easily verified that $\|\tilde{z}\| = \|\tilde{\phi}\| = \tilde{\phi}(\tilde{z}) = 1$.

Therefore the hypothesis about Q applies to the pair $\tilde{z}, \tilde{\phi}$: $\tilde{\phi}(Q(\tilde{z})) = \overline{\tilde{\phi}(c)}$.

This formula can be re-written as

$$\phi\begin{bmatrix} k_1 Q_1(z_1 \hat{c}_1, z_2 \hat{c}_2, z_3 \hat{c}_3) \\ k_2 Q_2(z_1 \hat{c}_1, z_2 \hat{c}_2, z_3 \hat{c}_3) \\ k_3 Q_3(z_1 \hat{c}_1, z_2 \hat{c}_2, z_3 \hat{c}_3) \end{bmatrix} = \overline{\phi\begin{bmatrix} k_1(c_1) \\ k_2(c_2) \\ k_3(c_3) \end{bmatrix}}.$$

But now we have

$$\left[w \mapsto \begin{bmatrix} k_1 Q_1(w_1 \hat{c}_1, w_2 \hat{c}_2, w_3 \hat{c}_3) \\ k_2 Q_2(w_1 \hat{c}_1, w_2 \hat{c}_2, w_3 \hat{c}_3) \\ k_3 Q_3(w_1 \hat{c}_1, w_2 \hat{c}_2, w_3 \hat{c}_3) \end{bmatrix} \right] \in \mathcal{L}_s^2(E)$$

and therefore

$$\begin{bmatrix} k_1(c_1) \\ k_2(c_2) \\ k_3(c_3) \end{bmatrix} \in E_0,$$

whence, by lemma 4, $k_2(c_2) = k_3(c_3) = 0$. It follows from our choice of k_2 and k_3 that $c_2 = 0$ and $c_3 = 0$, and the proof is complete. □

4. In this section we will discuss the case when the spaces involved in a direct sum are L^p-spaces.

We will denote by $\mathcal{M}$ the category of all Banach spaces $L^p(\Omega,\mu)$, where (Ω,μ) is a measure space, $p \in [1,\infty]$ and $\dim L^p(\Omega,\mu) \geq 1$; the morphisms in this category will be the linear isometries.

We define now another category $\mathcal{C}$ whose objects are the Banach spaces obtained from the objects of $\mathcal{M}$ by a finite number of operations of direct sum of the type $\oplus_r$ (with $r \in [1,\infty]$), and whose morphisms are the linear isometries. (If the collection $\mathcal{B}$ of all Banach spaces were a set, and not only a category, we would have defined $\mathcal{C}$ as the closure of $\mathcal{M}$ in $\mathcal{B}$ with respect to the operations $\oplus_r$).

We want to determine explicitly the orbit of the origin in the unit ball of an object of $\mathcal{C}$.

We introduce a notion we will need in the following: if E is an object in $\mathcal{C}$, E is linearly and topologically isomorphic to a product $F_1 \times \ldots \times F_k$ where $F_1, \ldots, F_k$ are objects of $\mathcal{M}$; for $i \in \{1, \ldots, k\}$ we can think of E as the space built up starting from F_i and adding to it other objects $G_1, \ldots, G_h$ of $\mathcal{C}$; that is, we can represent E by

$$(\ldots((F_i \oplus_{r_1} G_1) \oplus_{r_2} G_2) \oplus_{r_3} \ldots) \oplus_{r_h} G_h$$

(or by a similar formula where the sums are not all performed at the right side). In such a case we will say that «$\oplus_{r_1}, \ldots, \oplus_{r_h}$ are, in the order, the direct sums which appear in E *after* F_i». We briefly remark that, once an explicit representation of E is given, we can determine the direct sums appearing in E after F_i algorithmically (*i.e.* by a computer's program).

Before stating our theorem we record the following consequence of the results in [8, 9] and [1] and of lemmas 2 and 3:

THEOREM 7: if F is an object of $\mathcal{M}$, then $F_0 = F$ if F is either a Hilbert space or $F = L^\infty(\Omega,\mu)$, and $F_0 = \{0\}$ in all other cases.

THEOREM 8: let E be an object of $\mathcal{C}$, topologically and linearly isomorphic to a product of objects of $\mathcal{M}$, $F_1 \times \ldots \times F_k$. Then $E_0 = R_1 \times \ldots \times R_k$, where $R_i \subseteq F_i$ is either $\{0\}$ or the whole F_i.

Precisely, R_i is equal to F_i if, and only if, one of the following conditions is fulfilled:

(*a*) F_i is a Hilbert space and after F_i there are first some $\oplus_2$ (possibly none) and then some $\oplus_\infty$ (possibly none);

(*b*) $F_i = L^\infty(\Omega,\mu)$ and after F_i there are only $\oplus_\infty$ (possibly none).

PROOF: setting $R_i = F_i$ if one of the above conditions is fulfilled, and $R_i = \{0\}$ otherwise, it suffices to establish the following facts:

(I) $$E_0 \supseteq \{0\}_1 \times \ldots \times F_i \times \ldots \times \{0\}_k \quad \text{if } R_i = F_i,$$

(II) $$E_0 \subsetneq F_1 \times \ldots \times \{0\}_i \times \ldots \times F_k \qquad \text{if } R_i = \{0\}.$$

For the sake of simplicity we prove (I) and (II) when $i = 1$.

In the following we will assume that E is obtained from F_1 as

$$(\ldots((F_1 \oplus_{r_1} G_1) \oplus_{r_2} G_2) \oplus_{r_3} \ldots) \oplus_{r_h} G_h$$

(*i.e.* the direct sums appearing after F_1 are in the order $\oplus_{r_1}, \ldots, \oplus_{r_h}$).

We begin with (I). If F_1 satisfies condition (a) we have the following situation:

$$E = (F_1 \oplus_2 G_1 \oplus_2 \ldots \oplus_2 G_j) \oplus_\infty G_{j+1} \oplus_\infty \ldots \oplus_\infty G_h .$$

Hence, if we set $L_1 = G_1 \oplus_2 \ldots \oplus_2 G_j$ and $L_2 = G_{j+1} \oplus_\infty \ldots \oplus_\infty G_h$, we have $E = (F_1 \oplus_2 L_1) \oplus_\infty L_2$, (possibly $L_1 = \{0\}$ or $L_2 = \{0\}$).

Since F_1 is a Hilbert space, theorems 4 and 3 yield respectively:

$$(F_1 \oplus_2 L_1)_0 \supsetneq F_1 \times \{0\}, \qquad E_0 \supsetneq F_1 \times \{0\}_2 \times \ldots \times \{0\}_k .$$

If F_1 satisfies condition (b) we have the following situation: $E = F_1 \oplus_\infty G_1 \oplus_\infty \ldots \oplus_\infty G_h$; hence, if we set $L = G_1 \oplus_\infty \ldots \oplus_\infty G_h$, we have $E = F_1 \oplus_\infty L$ (possibly $L = \{0\}$). Since $F_1 = L^\infty(\Omega, \mu)$, it follows from theorem 7 that $(F_1)_0 = F_1$ and from theorem 3 we obtain $E_0 \supsetneq F_1 \times \{0\}_2 \times \ldots \times \{0\}_k$.

Now we prove (II). If neither condition (a) nor condition (b) is fulfilled, it is easy to check that at least one of the following conditions must hold:

(c) $F_1 = L^p(\Omega, \mu)$ with $p \notin \{2, \infty\}$ and $\dim F_1 \geq 2$;

(d) there is a $\oplus_p$ with $p \notin \{2, \infty\}$ after F_1;

(e) $F_1 = L^\infty(\Omega, \mu)$, $\dim F_1 \geq 2$ and the first direct sum appearing after F_1 is a $\oplus_2$;

(f) there is a $\oplus_\infty$ followed by a $\oplus_2$ after F_1.

We prove that (II) holds in all these cases.

(c) By theorem 7 we have $(F_1)_0 = \{0\}$. Moreover, theorem 1 implies that, no matter which direct sums are executed afterwards, no point in E_0 can have a non-zero F_1 coordinate, *i.e.* $E_0 \subsetneq \{0\} \times F_2 \times \ldots \times F_k$.

(d) If $r_j \notin \{2, \infty\}$ we set $L = (\ldots(F_1 \oplus_{r_1} G_1) \oplus_{r_2} \ldots) \oplus_{r_{j-1}} G_{j-1}$; theorem 2 implies that $(L \oplus_{r_j} G_j)_0 \subsetneq \{0\} \times G_j$ and we conclude as in case (c).

(e) F_1 is a commutative C^*-algebra with identity and $\dim F_1 \geq 2$, hence by theorem 5 we have $(F_1 \oplus_2 G_1)_0 \subsetneq \{0\} \times G_1$, and we conclude as in case (c) again.

(f) If $r_j = \infty$ and $r_{j+1} = 2$, setting $L = (\ldots(F_1 \oplus_{r_1} G_1) \oplus_{r_2} \ldots) \oplus_{r_{j-1}} G_{j-1}$, theorem 6 implies that $((L \oplus_\infty G_j) \oplus_2 G_{j+1})_0 \subsetneq \{0\} \times G_{j+1}$, and once again we conclude as in case (c).

The proof is complete. □

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