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Giuseppe De Cecco, Giuliana Palmieri

Length of curves on Lip manifolds

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Geometria differenziale. — *Length of curves on Lip manifolds.* Nota di GIUSEPPE DE CECCO e GIULIANA PALMIERI, presentata (*) dal Socio E. DE GIORGI.

ABSTRACT. — In this paper the length of a curve on a Lipschitz Riemannian manifold is defined. It is shown that the above definition is consistent with the definition of the geodesic distance already introduced by the authors, both in a geometrical and analytical way.

KEY WORDS: Lipschitz manifold; Length; Nonsmooth analysis.

RIASSUNTO. — *Lunghezza di curve su varietà di Lipschitz.* Su una varietà di Lipschitz dotata di metrica riemanniana di Lipschitz si introduce la nozione di lunghezza di una curva lipschitziana, mettendola in relazione con le distanze geodetiche introdotte (per via geometrica e per via analitica) in precedenti lavori dagli stessi autori.

The framework of the present paper is the class of the Lipschitz n-manifolds, generalizing, as well-known, the *PL manifolds* (*i.e.* the *polyedra*) and the *smooth manifolds*. They are well suited for an investigation of the differential properties that might fail to be true on sets of zero measure as emphasized by H. Whitney [8]. Moreover these manifolds can have vertices, edges, conical points, even not isolated.

Different authors have studied LIP manifolds from various points of view (*e.g.* [5], [7], [8]). Here we consider LIP manifolds (M, g) with a LIP metric g (following the presentation of N. Teleman [7]).

Recently in [1] and [2] we have defined intrinsic distances, that are induced by g and which agree with the usual one when M is smooth and g is a Riemannian metric or when M is a convex polyhedron or a graph of a LIP function imbedded in $\mathbb{R}^n$ (where g is the metric induced by the imbedding).

The main difficulties are the following: the Jacobian of the chart transformations is defined outside a n-dimensional null set and the components of g, assumed only to be measurable are defined up to an equivalence relation.

A first answer to the problem was given, in a geometric way, in [1]. There we have constructed a distance ρ using the length $L(\gamma)$ of LIP curves γ that are *transversal* to a null set $N \subset M$ (where *transversal* means $\text{mis}\{t \in [0,1];\ \gamma(t) \in N\} = 0$ and the measure $|N|$ of N is defined through g):

$$\rho(x,y) = \sup_N \{\inf L(\gamma);\ \gamma \in \mathcal{Lip}(I,\ M),\ \gamma(0) = x,\ \gamma(1) = y,\ \gamma\,\text{trans.}\,N,\ |N| = 0\}\,.$$

In [2] we have defined a distance δ in an integral way, overcoming the abovementioned difficulties:

$$(*)\qquad \delta(x,y) = \lim_{p\to\infty}\left[\inf\left\{\left(\int_M |du|^p\,dv\right)^{1/p};\ u \in \mathcal{Lip}(M),\ u(x) = 0,\ u(y) = 1\right\}\right]^{-1}$$

(*) Nella seduta del 10 marzo 1990.

where $|\cdot|$ and $d\nu$ depend, as usual on the metric g on M. We have proved that, under very general hypotheses, the two distances ρ and δ coincide.

In the present paper, which recalls notions and results of [1] and [2], we show that δ (and also ρ) is a *geodesic distance* for the metric space (M,δ), *i.e.* (M,δ) is a *length space* [4]. Then if (M,δ) is complete, the Hopf-Rinow theorem in the general form [6] ensures that two arbitrary points can be joined by a minimal geodesic.

Finally, we give a definition of length of a LIP curve that extends the classical one for Riemannian manifolds. More precisely we put

$$L(\gamma) = \sup_N \{\liminf_{\tau\to\gamma} L(\tau);\ \tau \in \mathcal{L}ip(I,M),\ \tau \,\text{trans.}\, N,\ |N| = 0\}$$

where the limit is with respect to uniform convergence.

We prove that also $L(\gamma)$, constructed with the LIP Riemannian structure g, coincides with the usual length $\mathcal{L}(\gamma)$, constructed with the distance δ.

Finally we mention that new conjectures about the distances that can suitably be introduced on LIP manifolds have been presented by E. De Giorgi in a recent conference [3] and developed in series of talks on this topic.

Precisely a notion of *quasi-Finslerian distance* has been put forward which is expressed by a formula similar to (*) and several conjectures on it have been formulated.

If some of these conjectures were to be confirmed, then the results of the present note and of [1] and [2] would have been extended to a wider framework.

In further studies, we intend to deal with these problems in the light of the results hitherto proved.

We thank E. De Giorgi for drawing our attention to this subject and for stimulating discussions.

1. Preliminaries

(1.1) A *Lipschitz manifold* (LIP *manifold*) of dimension n is a pair consisting of a topological manifold M and an equivalence class of LIP atlases [7]. A LIP atlas on M is a family of charts $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ $(\alpha \in \Lambda)$ where $\{U_\alpha\}$ forms an open cover of M, $\Phi_\alpha : U_\alpha \to V_\alpha$ maps homeomorphically U_α onto a set V_α which is open either in $\mathbb{R}^n$ or in $\mathbb{R}^n_+$ and $\forall \alpha, \beta$ $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}$ defines a Lipschitz homeomorphism.

(1.2) A *Riemannian metric* on M is a collection $g = \{g^\alpha\}$ where g^α is a Riemannian metric on $V_\alpha = \Phi_\alpha(U_\alpha) \subset \mathbb{R}^n$, with measurable components, that satisfy the compatibility conditions

$$\Phi^*_{\alpha\beta}(g^\alpha) = g^\beta \qquad \text{(no sum intended)}, \tag{1.3}$$

where the pull-back $\Phi^*_{\alpha\beta}$ is defined component-wise. The map $\Phi_{\alpha\beta}$ is LIP, then is differentiable *a.e.* and has bounded and measurable partial derivatives. Hence the hypotheses on g are admissible for any LIP manifold and in general one cannot ask for greater regularity.

(1.4) A Riemannian metric g will be called a LIP *Riemannian metric* on M if any g^α defines on V_α a L_2-norm which is equivalent to the standard L_2-norm, *i.e.* there should exist two positive constants k_α and K_α such that, for any smooth form ω with compact support in V_α

(1.5) $$k_\alpha \|\omega\|_E \leqslant \|\omega\|_{g^\alpha} \leqslant K_\alpha \|\omega\|_E .$$

It follows (by Theorem (3.1) of [1]) that is possible to find two strictly positive constants h_α and H_α, *s.t.* for every chart (U_α) and every n-ple $(v^1, \dots, v^n) \in \mathbb{R}^n$

(1.6) $$h_\alpha^2 \sum (v^i)^2 \leqslant \sum g_{ij}^\alpha(z)\, v^i v^j \leqslant H_\alpha \sum (v^i)^2 \quad a.e. \text{ in } z .$$

For simplicity we will write

$$\|v\|^2 = \sum (v^i)^2 , \qquad |v|_{(z)}^2 = \sum g_{ij}(z)\, v^i v^j .$$

In the following we shall always assume (unless otherwise indicated) M to be connected, paracompact and oriented and that $k = \inf k_\alpha$ is strictly positive. This last hypothesis is used only if $|M| = +\infty$, where

$$|M| = \text{mis}_n(M) = \int_M d\nu .$$

(1.7) A function u: $M \to \mathbb{R}^n$ is LIP if $\tilde{u} = u \circ \Phi_\alpha^{-1}$ is LIP for every chart. For such a function $|(du)(z)|_{g^\alpha}^2 = g_\alpha^{ij}(z)\, \partial_i \tilde{u}\, \partial_j \tilde{u}$ makes sense *a.e.* and it is easy to see that

$$\limsup_{y \to z} |u(y) - u(x)| / |y - z|_{(z)} = |(du)(z)|_{g^\alpha} .$$

(1.8) Every LIP atlas $\{(U_\alpha, \Phi_\alpha)\}$ defines a local metric consisting of pairs $\{U_\alpha, \sigma_\alpha\}$ where $\sigma_\alpha(x, y) = \|\Phi_\alpha(x) - \Phi_\alpha(y)\|$ is the euclidean distance on the charts. Moreover, equivalent LIP atlases define LIP equivalent local metrics, hence every LIP manifold can be regarded as a locally metric space.

Nay, by starting from the σ_α distances, a global distance σ can be constructed on M (which is connected) in the following way (See [5]).

Let $\{W_\alpha\}$ be a cover of M with $W_\alpha \subset U_\alpha$. For x, $y \in M$ let $\Pi(x,y)$ be the set of all finite sequences of the form $\pi = \{x_0, \dots, x_k; \alpha_1, \dots, \alpha_k\}$ such that $x_0 = x$, $x_k = y$ and $\{x_{j-1}, x_j\} \subset W_{\alpha_j}$. Then we set

(1.9) $$\sigma(x, y) = \inf \{ \textstyle\sum \sigma_{\alpha_j}(x_{j-1}, x_j)\, ; \quad j = 1, \dots, k; \quad \pi \in \Pi(x, y) \} .$$

The distance σ just introduced depends on the choice of charts; so there is no relation between σ and the Riemannian structure g. We will introduce, on the contrary, a distance, induced by g, which is intrinsic and coincides with the usual one in case (M, g) is a smooth Riemannian manifold.

2. Integral distance

(2.1) Let x, y be two distinct points of M and $\mathcal{Lip}(M; x, y) = \{u : M \to \mathbb{R} \text{ LIP}; u(x) = 0, u(y) = 1\}$. If $x = y$ we put $\delta(x, y) = 0$. If $x \neq y$ we put

(2.2) $$\delta(x, y) = \limsup_{p \to \infty} \left[\inf \left\{ \int_M |du|^p\, d\nu;\ u \in \mathcal{Lip}(M; x, y) \right\} \right]^{-1/p}$$

which of course depends on the metric g.

One sees that the limit exists and it is finite (since M is connected). Moreover $\delta: M \times M \to \mathbb{R}$ is a distance which is LIP equivalent to the distance σ induced on M by the charts. Hence the completeness with respect to σ implies the completeness with respect to δ.

(2.3) THEOREM. *Under our hypotheses on* M, *if* (M, δ) *is also complete, we have* $\delta(x,y)^{-1} = \min\{\|du\|_{L_\infty}; u \in \mathcal{Lip}(M; x, y)\}$. *Moreover if* $u \in \mathcal{Lip}(M)$, $|u(x) - u(y)| \leq \leq \|du\|_{L_\infty} \delta(x,y)$ i.e. u is LIP *also with respect to the metric* δ *of* M.

For the details and the proofs we refer to [2].

Notice that if (M, σ) is complete, then the expression of δ in (2.3) can be used in order to define the function δ.

3. Another distance

(3.1) As usual let M be a LIP manifold with LIP riemannian metric g. Set $\mathcal{N} = \{N \subset M; |N| = 0\}$ and denote by $\mathcal{Lip}_N(x,y;M) = \{\gamma \in \mathcal{Lip}(x,y;M);\ \text{mis}\{t \in I; \gamma(t) \cap N\} = 0\}$ the set of the *curves transversal* to N, which is not empty.

We recall that (see [1])

$$\mathcal{Lip}(x,y;M) = \{\gamma : [0,1] \to M \text{ LIP}; \gamma(0) = x, \gamma(1) = y\}.$$

As in [1], to which we refer for the details, if $\gamma \in \mathcal{Lip}_N(x,y;M)$ we introduce the integral

$$L_{\mathcal{A}}(\gamma) = \int_0^1 \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}\,dt = \int_0^1 |\dot{\gamma}|_{(\gamma(t))}\,dt \tag{3.2}$$

with the understanding that $L_{\mathcal{A}}(\gamma) = +\infty$ whenever it does not exist. However with a suitable choice of the set $N = N(\mathcal{A})$ of zero measure it is possible to give a meaning to (3.2). Then we call $L_{\mathcal{A}}(\gamma)$ *length of* γ (with respect to the atlas $\mathcal{A}$).

Since M is paracompact, we can choose a countable atlas $\mathcal{A}$. Let $\widetilde{D}_\alpha(\beta)$ be the subset of $\Phi_\alpha(U_\alpha \cap U_\beta)$ for which does not hold and observe that there exists a subset $E_\alpha \subset V_\alpha$, $|E_\alpha| = 0$, *s.t.* for every curve transversal to $\Phi^{-1}(E_\alpha)$, the functions $g_{ij}^\alpha \circ \Phi_\alpha \circ \gamma$ are measurable $\forall\, i,j$. Set $F_\alpha = (\bigcup_\beta \widetilde{D}_\alpha(\beta)) \cup E_\alpha$. If $N \supseteq \bigcup_\alpha \Phi_\alpha^{-1}(F_\alpha)$, then the integral (3.2) makes sense.

Set

$$\rho_N(x,y) = \inf\{L_{\mathcal{A}}(\gamma); \gamma \in \mathcal{Lip}_N(x,y;M)\}; \tag{3.3}$$

then $\rho(x,y) = \sup\{\rho_N(x,y); N \in \mathcal{N}\}$ is a *distance* on M independent of $\mathcal{A}$.

Remark that if (M, g) is a Riemannian *smooth* manifold, then ρ agrees with the intrinsic distance induced by g.

Moreover the following theorem holds (see [2, Theorems (6.11) and (7.1)]).

(3.4) THEOREM. *If* (M, g) *is a* LIP *manifold, then*

$$\|d\rho(x,\cdot)\|_{L_\infty} = 1\,, \qquad \|d\delta(x,\cdot)\|_{L_\infty} = 1\,.$$

Moreover $\delta = \rho$.

(3.5) Now $(M,\rho)=(M,\delta)$ is a metric space and it is possible to define in the usual way the length of a curve γ, *i.e.*

$$\mathscr{L}(\gamma)=\sup_T\left\{\sum_i \rho(\gamma(t_i),\gamma(t_{i+1}));\ T \text{ decomposition of } I\right\}.$$

In the following we shall show that ρ is a *geodesic distance.*

First we note that

(3.6) LEMMA. *For every $x,y\in M$ there exists $N\in\mathcal{N}$ s.t. $\rho_N(x,y)=\rho(x,y)$.*

Now if C_α is a countable set and dense in V_α, then by the lemma $N_\alpha\in\mathcal{N}$ exists *s.t.* $\rho_{N_\alpha}(x,y)=\rho(x,y)$. If

$$N=N(\mathcal{A})=\bigcup_{\alpha\in\mathrm{N}}\Phi_\alpha^{-1}(N_\alpha\cup F_\alpha) \tag{3.7}$$

then $N\in\mathcal{N}$ and $\mathscr{Lip}_N(x,y;M)\neq\emptyset$.

(3.8) THEOREM. *If N is the set in* (3.7) *and $\gamma\in\mathscr{Lip}_N(x,y;M)$, then $\mathscr{L}(\gamma)\leq L_{\mathcal{A}}(\gamma)$.*

PROOF. The proof follows easily from

$$\rho(\gamma(t_i),\gamma(t_{i+1}))\leq\int_{t_i}^{t_{i+1}}|\dot\gamma|\,dt. \tag{3.9}$$

Let $x_i=\gamma(t_i)$. If $x_i,\ x_{i+1}\in C=\bigcup_\alpha C_\alpha$ the claim is trivial. If $x_k\notin C$, by density of C, for every $\varepsilon>0$ there exist $\tilde x_k\in C$, *s.t.* $\rho(x_k,\tilde x_k)<\varepsilon$, and a curve $\tau_k\in\mathscr{Lip}_N(x_k,\tilde x_k;M)$ *s.t.* $L_{\mathcal{A}}(\tau_k)<\varepsilon$. Now $\tilde\gamma=\gamma\mid[t_i,t_{i+1}]\cup\tau_i\cup\tau_{i+1}\in\mathscr{Lip}_N(\tilde x_i,\tilde x_{i+1};M)$ and $\rho(x_i,x_{i+1})\leq$ $\leq L_{\mathcal{A}}(\gamma\mid[t_i,t_{i+1}])+4\varepsilon$ from which the theorem follows. ■

(3.10) THEOREM. $\rho(x,y)=\inf\{\mathscr{L}(\gamma);\gamma\in\mathscr{Lip}(x,y;M)\}$.

PROOF. By the definition of ρ and by lemma (3.6), $\gamma_\varepsilon\in\mathscr{Lip}_N(x,y;M)$ exists *s.t.* $\rho(x,y)=\rho_N(x,y)\leq L_{\mathcal{A}}(\gamma_\varepsilon)\leq\rho_N(x,y)+\varepsilon=\rho(x,y)+\varepsilon$ from which, by theorem (3.8), $\mathscr{L}(\gamma_\varepsilon)\leq L_{\mathcal{A}}(\gamma_\varepsilon)\leq\rho(x,y)+\varepsilon$ hence $\inf\{\mathscr{L}(\gamma)\}\leq\rho(x,y)$. The opposite inequality is obtained if $T=\{0,1\}$ is the chosen decomposition of I in (3.9). ■

Now it is possible to extend the theorem (3.4):

(3.11) THEOREM. *If M is a paracompact oriented* LIP *manifold, without further hypotheses, there exists*

$$\lim_{p\to+\infty}[\inf\{\|du\|_p;u\in\mathscr{Lip}(M;x,y)\}]=\delta^{-1}(x,y)$$

and $\delta^{-1}(x,y)=\min\{\|du\|_{L_\infty};u\in\mathscr{Lip}(M;x,y)\}$. Moreover $\delta(x,y)=\rho(x,y)$.

PROOF. By theorem (3.10), (M,ρ) is a length space so that the closed balls are compact. The proof of the existence of the previous limit is like that the theorem (2.6) of [2], if one replace σ by ρ and notices that $\|d\rho(x,\cdot)\|_\infty\leq 1$. For every $Q\subset M$ with $|Q|<+\infty$ we put

$$b_p(Q)=\inf\left\{\left(\int_Q|du|^p\,dv\right)^{1/p};u\in\mathscr{Lip}(M;x,y)\right\}.$$

Then

$$\delta^{-1}(x,y)=\sup_Q\left\{\lim_{p\to\infty}b_p(Q)\right\}.$$

But $b_p(Q) \leqslant \inf\{\|du\|_\infty |Q|^{1/p}\} \leqslant \rho^{-1}(x,y)|Q|^{1/p}$, then, by passing to the limit, $\delta^{-1}(x,y) \leqslant \inf\{\|du\|_\infty\} \leqslant \rho^{-1}(x,y) \Rightarrow \rho(x,y) \leqslant \delta(x,y)$. As in [2, Theorem (6.10)] we have $\|d\delta(x,\cdot)\|_\infty = 1$ and hence $\delta^{-1}(x,y) = \min\{\|du\|_\infty ; u \in \mathcal{Lip}(M; x, y)\}$. Let $N(\delta)$ be the set of the points $y \in M$ when either $d\delta(x,y)$ is not defined or $|d\delta(x,y)| > 1$; then for every $\varepsilon > 0$ there exists a curve y transversal to $N(\delta)$ and *s.t.* $L(\gamma) \leqslant \rho + \varepsilon$. Therefore

$$\delta(x,y) = \int_0^1 \frac{d}{dt}\delta(x,\gamma(t))\,dt \leqslant \rho(x,y) + \varepsilon$$

from which $\delta(x,y) = \rho(x,y)$. ■

4. Length of a LIP curve

(4.1) Let (M, g) a LIP manifold with a LIP Riemannian metric g and let $L_{\mathcal{A}}(\gamma)$ be as in (3.1). Moreover on the space of the curves on M we consider the topology of uniform convergence induced by a distance δ.

If $\gamma \in \mathcal{Lip}(x, y; M)$ we put

(4.2) $$L(\gamma) = \sup_N \{\liminf_{\tau \to \gamma} L_{\mathcal{A}}(\tau); \tau \in \mathcal{Lip}_N(x, y; M), N \in \mathcal{N}\}.$$

By an argument analogous to [1, Theorem (4.4)] one sees that $L(\gamma)$ is independent of the atlas; thus $L(\gamma)$ is called *length of the curve* γ (with respect to Riemannian metric g and distance δ).

(4.3) Theorem. *If* δ is the integral distance (3.2) *(or the distance* ρ*)*, then $\mathcal{L}(\gamma) = L(\gamma)$, $\gamma \in \mathcal{Lip}(x, y; M)$.

Proof. (1) $\mathcal{L}(\gamma) \leqslant L(\gamma)$ - As in [1, Theorem (4.4)], if $N(\mathcal{A})$ is the set defined in (3.7), then

$$L(\gamma) = \sup_E \{\liminf_{\tau \to \gamma} L_{\mathcal{A}}(\tau); \tau \text{ trans. } E \cup N(\mathcal{A}), E \in \mathcal{N}\}$$

So we can find a sequence of LIP curves $\{\tau_m\}$, transversal to $N(\mathcal{A})$ and convergent to γ, *s.t.*

$$\lim_{m \to \infty} L_{\mathcal{A}}(\tau_m) = L(\gamma);$$

now $\mathcal{L}(\tau_m) \leqslant L_{\mathcal{A}}(\tau_m)$ by Theorem (3.8) and by the lower semicontinuity of $\mathcal{L}$ [6] the claim is obtained.

(2) $L(\gamma) \leqslant \mathcal{L}(\gamma)$ - By definition of $\mathcal{L}(\gamma)$ and the continuity of γ, fixed $\varepsilon > 0$, there exists a decomposition $\{0 = t_0 < t_1 < \ldots < t_k = 1\}$ *s.t.*

(4.4) $$\mathcal{L}(\gamma) - \varepsilon \leqslant \sum \rho(x_{i-1}, x_i) \leqslant \mathcal{L}(\gamma), \qquad x_i = \gamma(t_i),$$

(4.5) $$\rho(x_{i-1}, \gamma(t)) < \varepsilon \qquad \forall t \in [t_{i-1}, t_i].$$

Moreover let $\tau_i : [t_{i-1}, t_i] \to M$ be a LIP curve joining x_{i-1} with x_i and transversal to $E \supset N(\mathcal{A})$ *s.t.*

(4.6) $$\rho(x_{i-1}, x_i) \leqslant L(\tau_i) \leqslant \rho(x_{i-1}, x_i) + \varepsilon/k.$$

Set $\tau_\varepsilon = \bigcup_i \tau_i$, which is transversal to $N(\mathcal{A})$; then by (4.4) and (4.6) it follows

(4.7) $$\mathcal{L}(\gamma) - \varepsilon \leqslant L(\tau_\varepsilon) \leqslant \mathcal{L}(\gamma) + \varepsilon.$$

To get the conclusion it is enough to prove that $\tau_\varepsilon \to \gamma$ for $\varepsilon \to 0$. Indeed by (3.9)

$$\rho(\tau_i(t), x_i) \leq \int_{t_i}^{t} |\dot{\tau}_i|\, dt \leq \int_{t_{i-1}}^{t_i} |\dot{\tau}_i|\, dt$$

then by (4.4) and (4.6) $\rho(\tau_i(t), \gamma(t)) \leq \rho(\tau_i(t), x_{i-1}) + \rho(x_{i-1}, \gamma(t)) < \varepsilon(1 + 1/k) + \varepsilon$.

Finally by definition of L it follows

$$L(\gamma) \leq \sup_N \{\lim_{\varepsilon \to 0} L(\tau_\varepsilon);\, \tau_\varepsilon \text{ trans. } N\} \leq \mathscr{L}(\gamma)\,. \quad \blacksquare$$

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G. De Cecco: Dipartimento di Matematica
Università degli Studi di Lecce
Via per Arnesano - 73100 Lecce

G. Palmieri: Dipartimento di Matematica
Università degli Studi di Bari
Traversa 200 Re David, 4 - 70125 Bari