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On global controllability of linear time dependent control systems

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Equazioni differenziali ordinarie. — *On global controllability of linear time dependent control systems.* Nota (*) di ALBERTO TONOLO, presentata dal Corrisp. R. CONTI.

ABSTRACT. — Let (A, B) be a linear time dependent control process, defined on an open interval $J =]\alpha, \omega[$ with $\alpha \geqslant -\infty$ and $\omega \leqslant \infty$; in this paper we give a description of the function $\tau: I \to J$, $\tau(t) = \inf\{t' > t: (A, B)$ is $[t, t']$-globally controllable from $0\}$ where $I = \{t \in J: \exists t' \in J$ with (A, B) $[t, t']$-globally controllable from $0\}$.

KEY WORDS: Linear control process; Global controllability; Kalman's matrix.

RIASSUNTO. — *Sulla controllabilità globale dei sistemi di controllo lineari dipendenti dal tempo.* Sia (A, B) un processo di controllo lineare dipendente dal tempo, definito su un intervallo aperto $J =]\alpha, \omega[$ con $\alpha \geqslant -\infty$ e $\omega \leqslant \infty$; in questo lavoro diamo una descrizione della funzione $\tau: I \to J$, $\tau(t) = \inf\{t' > t: (A, B)$ è $[t, t']$-globalmente controllabile da $0\}$ dove $I = \{t \in J: \exists t' \in J$ con (A, B) $[t, t']$-globalmente controllabile da $0\}$.

0. INTRODUCTION

0.1. Let us consider the linear control process

$$(A, B) \qquad x' = A(t)\,x + B(t)\,u(t)$$

where, as usual, $x = x(t)$ is an n-vector, $u(t)$ an m-vector, $A(t)$ and $B(t)$ are real matrices of type $n \times n$, $n \times m$ respectively, both depending on t which varies on some interval $J =]\alpha, \omega[$ with $\alpha \geqslant -\infty$ and $\omega \leqslant \infty$. We suppose $A, B \in L^1_{\text{loc}}(J)$ and $u \in U_B = \{u \in L^1_{\text{loc}}(J): Bu \in L^1_{\text{loc}}(J)\}$. For each $t_o \in J$, $v \in \mathbf{R}^n$ and $u \in U_B$ the solution $t \to x(t, t_0, v, u)$ of (A, B) such that $x(t_0, t_0, v, u) = v$ is represented by

$$x(t, t_0, v, u) = E(t, t_0)\,v + \int_{t_0}^{t} E(t, s)\,B(s)\,u(s)\,ds$$

where E is the transition matrix of $dx/dt - A(t)\,x = 0$.

0.2. DEFINITION. Given $S, T \in J$, $S < T$, we say that (A, B) is *globally* $[S, T]$*-controllable from zero* (in short $[S, T]$-g.c.) by means of a set $U \subseteq U_B$ if and only if for every $w \in \mathbf{R}^n$ there exists $u \in U$ such that $x(S, T, 0, u) = w$.

0.3. Given a linear control process (A, B), let I be the set of $t \in J$ for which there exists $t' \in J$ such that (A, B) is $[t, t']$-g.c. by means of U; clearly, if $I \neq \emptyset$, $I =]\alpha, \hat{\omega}[$ with $\hat{\omega} \leqslant \omega$. The purpose of this paper is to study the function

$$\tau: I \to J, \qquad \tau(t) = \inf\{t' > t: (A, B) \text{ is } [t, t']\text{-g.c.}\}.$$

We have achieved the following results:

(*) Pervenuta all'Accademia il 7 settembre 1990.

PROPOSITION (Prop. 1.1). *Let* $B \in L^1_{\mathrm{loc}}(J)$, $U = L^\infty_{\mathrm{loc}}(J)$, *then* $\tau: I \to J$ *is non decreasing and continuous on the right.*

MAIN THEOREM (Th. 1.5). *Let* $B \in L^2_{\mathrm{loc}}(J)$, $U = L^2_{\mathrm{loc}}(J)$, *then* $\tau: I \to J$ *is piecewise linear; more precisely it is piecewise constant, piecewise the identity.*

In the last section we will give some meaningful examples of this function.

1. THE FUNCTION τ

1.1. PROPOSITION. *Let* $B \in L^1_{\mathrm{loc}}(J)$, $U = L^\infty_{\mathrm{loc}}(J)$, *then* $\tau: I \to J$ *is non decreasing and continuous on the right.*

PROOF. Let $r \leqslant s < t \in J$, if (A, B) is $[s, t]$-g.c., then obviously (A, B) is $[r, t]$-g.c.; so $\tau: I \to J$ is non decreasing. Next, fixed $\varepsilon > 0$, $t \in I$ we consider $\tau(t) + \varepsilon$; being (A, B) $[t, \tau(t) + \varepsilon]$-g.c., there exists, by Th. 7.4.4 of [1], $\theta > 0$ such that (A, B) is $[t + \theta, \tau(t) + \varepsilon - \theta]$-g.c.; then for each $s \in [t, t + \theta]$ we have $\tau(s) \in [\tau(t), \tau(t) + \varepsilon]$.

1.2. In the sequel we assume $B \in L^2_{\mathrm{loc}}(J)$, $U = L^2_{\mathrm{loc}}(J)$; under these hypotheses we can apply the Kalman's criterion for which (A, B) is $[S, T]$-g.c. if and only if the matrix

$$K(S, T) = \int_S^T E(T, r)\, B(r)\, B^*(r)\, E^*(T, r)\, dr$$

is positive definite (see [2]).

Therefore $\tau(t_0) = \inf\{t > t_0 : K(t_0, t) \text{ is positive definite}\}$.

1.3. LEMMA. *Let A and $B(\alpha)$ be two symmetric matrices, positive semi-definite, such that $y^* A y = 0$ implies $y^* B(\alpha) y = 0$ and $\lim_{\alpha \to 0} B(\alpha) = 0$. There exists $\hat{\alpha} > 0$ such that for each $\alpha \leqslant \hat{\alpha}$ we have that $x^* A x > 0$ implies $x^* B(\alpha) x < x^* A x$.*

PROOF. We can choose a base $e_1, e_2, e_3, \ldots, e_n$ of $\mathbf{R}^n$ such that the matrix of the bilinear form induced by A is the diagonal matrix

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

We continue to write A and $B(\alpha)$ for the new matrices. Set $B(\alpha) = (\beta_{ij})_{i,j}$, we have, by the hypothesis, $\beta_{n-i,n-i} = e^*_{n-i} B(\alpha)\, e_{n-i} = 0$ for $i = 0, \ldots, m-1$. Next, being $B(\alpha)$

symmetric and semi-positive definite, it results $\beta_{j,n-i} = 0$ for $i = 0, \dots, m-1$; in fact, if $i = 0, \dots, m-1$, the determinant of the principal minor made on j-th and $(n-i)$-th rows and columns is $-\beta^2_{j,n-1}$. So we have

$$B(\alpha) = \begin{bmatrix} \beta_{11} \cdots\cdots \beta_{1m} & \\ \cdots\cdots\cdots & 0 \\ \beta_{m1} \cdots\cdots \beta_{mm} & \\ 0 & 0 \end{bmatrix}.$$

Let us consider an arbitrary vector x of $\mathbf{R}^n$ written in spheric coordinates:
$x = (\rho \sin\phi_1 \sin\phi_2 \dots \sin\phi_{n-1}, \rho\cos\phi_1 \sin\phi_2 \dots \sin\phi_{n-1}, \dots, \rho\cos\phi_{n-2} \sin\phi_{n-1}, \rho\cos\phi_{n-1})$.
It's easy to verify that $x^* Ax = \rho^2 \sin^2\phi_m \dots \sin^2\phi_{n-1}$ and

$$x^* B(\alpha)\, x = \sum_{i,j=1}^{n} \beta_{ij}(\alpha)\, x_i x_j = \sum_{i,j=1}^{m} \beta_{ij}(\alpha)\, x_i x_j \leqslant (\rho^2 \sin^2\phi_m \dots \sin^2\phi_{n-1}) \sum_{i,j=1}^{m} |\beta_{ij}(\alpha)|\,.$$

Since $\lim_{\alpha\to 0} B(\alpha) = 0$, there exists $\hat{\alpha} > 0$ such that for each $0 \leqslant \alpha \leqslant \hat{\alpha}$ we have

$$\sum_{i,j=1}^{m} |\beta_{ij}(\alpha)| < 1\,.$$

Then if $x^* Ax = \rho^2 \sin^2\phi_m \dots \sin^2\phi_{n-1} \neq 0$, for each $\alpha \leqslant \hat{\alpha}$ we have $x^* B(\alpha)\, x < x^* Ax$.

1.4 Lemma. *We have the equality*

$$K(t_0 + \alpha, t_1 + \varepsilon) = K(t_1, t_1 + \varepsilon) +$$
$$+ E(t_1 + \varepsilon, t_1)[K(t_0, t_1) - E(t_1, t_0 + \alpha)\, K(t_0, t_0 + \alpha)\, E^*(t_1, t_0 + \alpha)]\, E^*(t_1 + \varepsilon, t_1).$$

Proof. It follows immediately from the «group property» of the transition matrix E.

1.5 Main Theorem. *Let $B \in L^2_{\mathrm{loc}}(J)$, $U = L^2_{\mathrm{loc}}(J)$, then τ is piecewise linear; more precisely it is piecewise constant, piecewise the identity.*

Proof. Let $t_0 \in I$, we will prove that if $t_1 = \tau(t_0) > t_0$ then there exists $\hat{\alpha} > 0$ such that for each $s \in [t_0, t_0 + \hat{\alpha}]$ it is $\tau(s) = \tau(t_0)$. It will be sufficient to verify that $x \neq 0$ and $\varepsilon > 0$ imply $x^* K(t_0 + \alpha, t_1 + \varepsilon)\, x > 0$ for each $\alpha \leqslant \hat{\alpha}$. By Lemma 1.4 we have

$$(\#) \qquad x^* K(t_0 + \alpha, t_1 + \varepsilon)\, x = x^* K(t_1, t_1 + \varepsilon)\, x +$$
$$+ x^* E(t_1 + \varepsilon, t_1)[K(t_0, t_1) - E(t_1, t_0 + \alpha)\, K(t_0, t_0 + \alpha)\, E^*(t_1, t_0 + \alpha)]\, E^*(t_1 + \varepsilon, t_1)\, x.$$

a) Being the Kalman's matrix always positive semi-definite, by (#) with $\varepsilon = 0$, it results that if $x^* K(t_0, t_1)\, x = 0$ then $x^* E(t_1, t_0 + \alpha)\, K(t_0, t_0 + \alpha)\, E^*(t_1, t_0 + \alpha)\, x = 0$, for $0 \leqslant x^* K(t_0 + \alpha, t_1)\, x = -x^* E(t_1, t_0 + \alpha)\, K(t_0, t_0 + \alpha)\, E^*(t_1, t_0 + \alpha)\, x \leqslant 0$;

b) Since for each $\varepsilon > 0$ (A, B) is $[t_0, t_1 + \varepsilon]$-g.c., then $K(t_0, t_1 + \varepsilon)$ is positive definite; let y be such that $[y^* E(t_1 + \varepsilon, t_1)]\, K(t_0, t_1)[E^*(t_1 + \varepsilon, t_1)\, y] = 0$, then by *a*) $[y^* E(t_1 + \varepsilon, t_1)]\, E(t_1, t_0 + \alpha)\, K(t_0, t_0 + \alpha)\, E^*(t_1, t_0 + \alpha)[E^*(t_1 + \varepsilon, t_1)\, y] = 0$ and so in this case $y^* K(t_0 + \alpha, t_1 + \varepsilon)\, y = y^* K(t_1, t_1 + \varepsilon)\, y$ for each $\alpha \geqslant 0$; now if $\alpha = 0$ we have $0 < y^* K(t_0, t_1 + \varepsilon)\, y$ and so for each $y \neq 0$ it results $y^* K(t_1, t_1 + \varepsilon)\, y > 0$. Then for each $\alpha \geqslant 0$, $y \neq 0$ implies $y^* K(t_0 + \alpha, t_1 + \varepsilon)\, y > 0$.

c) Next let y be such that
$$[y^* E(t_1+\varepsilon, t_1)]\, K(t_0, t_1)[E^*(t_1+\varepsilon, t_1)\, y] > 0;$$
it is easy to see that $A = K(t_0, t_1)$ and $B(\alpha) = E(t_1, t_0+\alpha)\, K(t_0, t_0+\alpha)\, E^*(t_1, t_0+\alpha)$ satisfy the hypothesis of Lemma 1.3; hence there exists $\hat{\alpha} > 0$ such that for each $0 \leqslant \alpha \leqslant \hat{\alpha}$ $[y^*\ E(t_1+\varepsilon, t_1)]\ E(t_1, t_0+\alpha)\ K(t_0, t_0+\alpha)\ E^*(t_1, t_0+\alpha)\ [E^*(t_1+\varepsilon, t_1)y] < [y^*\ E(t_1+\varepsilon, t_1)]\cdot$ $\cdot K(t_0, t_1)[E^*(t_1+\varepsilon, t_1)\, y]$ and so, being $y^* K(t_1, t_1+\varepsilon)\, y \geqslant 0$, for each $0 \leqslant \alpha \leqslant \hat{\alpha}$ we have $y^* K(t_0+\alpha, t_1+\varepsilon)\, y > 0$.

Finally, by *b*) and *c*), the matrix $K(t_0+\alpha, t_1+\varepsilon)$ results positive definite for each $\alpha \leqslant \hat{\alpha}$, and then, for the arbitrariness of ε, for each $s \in [t_0, t_0+\hat{\alpha}]$ we have $\tau(s) = t_1 = \tau(t_0)$.

2. Examples

2.1. In this section we will give two examples of linear control process for which the associated functions τ are respectively one piecewise constant, the other alternatively piecewise constant and piecewise the identity.

2.2. Let
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$B(t) = \begin{cases} \begin{bmatrix} 1/2 - (t-[t]) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t-[t] \leqslant 1/2, \\ \begin{bmatrix} 0 & (t-[t]-1/2)(t-[t]-1) \\ 0 & t-[t]-1/2 \end{bmatrix} & \text{if } t-[t] \geqslant 1/2. \end{cases}$$
Since
$$E(T,t) = I + (T-t)\, A = \begin{bmatrix} 1 & T-t \\ 0 & 1 \end{bmatrix}$$
it follows
$$E(T,t)\, B(t) = \begin{cases} \begin{bmatrix} 1/2 - (t-[t]) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t-[t] \leqslant 1/2, \\ \begin{bmatrix} 0 & (t-[t]-1/2)(T-[t]-1) \\ 0 & t-[t]-1/2 \end{bmatrix} & \text{if } t-[t] \geqslant 1/2. \end{cases}$$
Hence
$$y^* E(T,t)\, B(t) = \begin{cases} ((1/2-(t-[t]))\, y_1, 0) & \text{if } t-[t] \leqslant 1/2, \\ (0, (t-[t]-1/2)((T-[t]-1)\, y_1 + y_2)) & \text{if } t-[t] \geqslant 1/2. \end{cases}$$
Therefore (A,B) is $[S,T]$-g.c., with $S<T$, if and only if $1/2 \in \{t-[t] : t \in]S,T[\}$ or $0 \in \{t-[t] : t \in]S,T[\}$. From this it follows that $I = \mathbf{R}$ and for each $t \in \mathbf{R}$
$$\tau(t) = \begin{cases} [t] + 1/2 & \text{if } t-[t] < 1/2, \\ [t+1] & \text{if } t-[t] \geqslant 1/2. \end{cases}$$

2.3. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B(t) = \begin{cases} \begin{bmatrix} 1/2 - (t - [t]) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t - [t] \leqslant 1/2, \\ \begin{bmatrix} 0 & (t - [t] - 1/2)(t - [t] - 1) \\ (t - [t] - 1/2)(t - [t] - 1) & 0 \end{bmatrix} & \text{if } t - [t] \geqslant 1/2. \end{cases}$$

Since

$$E(T,t) = I + (T - t)A = \begin{bmatrix} 1 & T - t \\ 0 & 1 \end{bmatrix}$$

it follows

$$E(T,t)\,B(t) =$$

$$= \begin{cases} \begin{bmatrix} 1/2 - (t - [t]) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t - [t] \leqslant 1/2, \\ \begin{bmatrix} (T - t)(t - [t] - 1/2)(t - [t] - 1) & (t - [t] - 1/2)(t - [t] - 1) \\ (t - [t] - 1/2)(t - [t] - 1) & 0 \end{bmatrix} & \text{if } t - [t] \geqslant 1/2. \end{cases}$$

Hence

$$y^* E(T,t)\,B(t) =$$

$$= \begin{cases} ((1/2 - (t - [t]))\,y_1, 0) & \text{if } t - [t] \leqslant 1/2, \\ (((T - t)\,y_1 + y_2)(t - [t] - 1/2)(t - [t] - 1), (t - [t] - 1/2)((t - [t] - 1)\,y_1)) & \\ & \text{if } t - [t] \geqslant 1/2. \end{cases}$$

Therefore (A,B) is $[S,T]$-g.c., with $S < T$, if and only if $1/2 \in \{t - [t] : t \in]S,T[\}$ or $S - [S] \in [1/2, 1[$. From this it follows that $I = \mathbf{R}$ and for each $t \in \mathbf{R}$

$$\tau(t) = \begin{cases} t & \text{if } t - [t] \in [1/2, 1[, \\ [t] + 1/2 & \text{otherwise}. \end{cases}$$

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