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Geometria differenziale. — *On the spectrum of Riemannian submersions with totally geodesic fibers.* Nota (*) di GÉRARD BESSON e MANLIO BORDONI, presentata dal Socio E. MARTINELLI.

ABSTRACT. — In this *Note* we give a rule to compute explicitely the spectrum and the eigenfunctions of the total space of a Riemannian submersion with totally geodesic fibers, in terms of the spectra and eigenfunctions of the typical fiber and any associated principal bundle.

KEY WORDS: Laplace-Beltrami operator; Riemannian submersion; Representations of Lie groups.

RIASSUNTO. — *Sullo spettro di submersioni riemanniane a fibre totalmente geodetiche.* In questa *Nota* diamo una regola di calcolo esplicito dello spettro e delle autofunzioni dello spazio totale di una submersione riemanniana a fibre totalmente geodetiche, in termini dello spettro e delle autofunzioni della fibra tipo e di un qualsiasi fibrato principale associato.

1. INTRODUCTION

Our purpose by this work is to give an explicit method to compute the spectrum (eigenfunctions and eigenvalues with their multiplicity) of the Laplace-Beltrami operator Δ^M on the total space M of a Riemannian submersion with totally geodesic fibers.

It is well known the interest in Riemannian submersions. L. Bérard Bergery and J. P. Bourguignon introduce (see [1]) a decomposition of Δ^M as a sum of two (non Laplacian) operators, vertical Δ_v^M and horizontal Δ_h^M, corresponding to the natural decomposition of the metric.

As these two operators commute in the case in point, the eigenvalues of Δ^M are sums of eigenvalues of each of them. Not all the sums are admissible, the choice depending on the global geometry of the submersion.

Here we consider the principal bundle P canonically associated to the submersion $\pi: M \to B$. Based on the linear representations of the structure group G of the bundle and on the decomposition $\Delta^P = \Delta_v^P + \Delta_h^P$, we can determine which eigenvalues are in fact involved in the above sums and construct an explicit Hilbertian basis of eigenfunctions of Δ^M (see Theorem, sect. 3).

The result could be used, for instance, to construct other examples (besides those already known) of isospectral and non isometric manifolds, by choosing two such manifolds as fibers of suitable submersions.

2. THE GEOMETRY OF THE SITUATION

Let M and B be two compact connected Riemannian manifolds and

$$\pi: M \to B$$

a Riemannian submersion. We furthermore assume that the fibers of π are totally

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geodesic submanifolds. It is known (see [6]) that they are all isometric, when endowed with the metric induced from M; therefore we can choose one of them to be *the typical fiber which we shall call* F.

The general theory (see [7], p. 55) says that in this situation the fibration π is associated to a principal bundle $p: P \to B$ whose structure group G can be chosen to be the (compact) Lie group of isometries of F.

In this article we intend to use extensively the correspondance between the two bundles, keeping in mind that different Riemannian submersions with totally geodesic fibers can be associated to the same principal bundle.

So the situation can be sommarized in the following commutative diagram:

$$\begin{tikzcd} P \times F \arrow[r, "\omega"] \arrow[d, "pr_1"'] & M \arrow[d, "\pi"] \\ P \arrow[r, "p"'] & B \end{tikzcd}$$

where pr_1 is the projection onto the first factor of the product and ω the quotient of $P \times F$ by the diagonal action of G, $M = P \times_G F$.

By a standard result (see [10]) we know that for any chosen G-invariant metric on G there exists a unique metric on P such that the map p is a Riemannian submersion with totally geodesic fibers isometric to G. In the sequel we will always choose bi-invariant metrics on G.

Now it is worthy emphasizing that if $P \times F$ is endowed with the product metric, the map ω is *not* in general a Riemannian submersion (see [4, 2]). To circumvent this difficulty, we shall proceed as follows.

Let $g_P(t)$ be the metric on P given by the J. Vilms' theorem which restricted to the fibers is the bi-invariant metric with total mass $t^{(\dim G)}$ (the canonical variation according to [1]).

Now in J. Cheeger [4] it is showed that there exists a metric $g_F(t)$ depending analytically on t^{-2}, invariant by the action of G on F, and such that:

i) for t large enough, if $P \times F$ is endowed with the metric $g_P(t) \oplus g_F(t)$, then ω is a Riemannian submersion;

ii) when t goes to infinity then $g_F(t)$ goes to the original metric on F;

iii) the fibers of ω (which are homogeneous submanifolds) have a mean curvature vector $H(t)$ which is analytic in t and converges to zero when t goes to infinity (see also [2]).

For the sake of simplicity we shall write $P_t \times F_t$ instead of $(P \times F, g_P(t) \oplus g_F(t))$ and we shall set $F_\infty = F$.

The Laplacians. For the Laplacian on the total space of a Riemannian submersion

$X \to B$, one has the decomposition (see [1]) $\Delta^X = \Delta^X_v + \Delta^X_h$ where Δ^X_v, Δ^X_h (although not Laplacians) we shall call, according to [1]:

$$\Delta^X_h = \text{horizontal Laplacian on } X, \qquad \Delta^X_v = \text{vertical Laplacian on } X.$$

Furthermore if $(\tilde{Z}_i)$, $i = 1, \ldots, \dim B$, is an orthonormal (local) moving frame on B and (Z_i) are the corresponding basic vector fields on X (see [8]), *i.e.* the fields which are pulled back from B, one has the formula

$$\Delta^X_h = -\sum_{i=1}^{\dim B} (Z_i \circ Z_i - D_{Z_i} Z_i) + H$$

where D is the Levi-Civita connection on X and H the mean curvature vector of the fiber.

If the fibers are totally geodesic then $H = 0$ and the two operators Δ^X_v and Δ^X_h commute and thus are simultaneously diagonalizable.

For $X = P_t$ one has (see [1]): $\Delta^{P_t}_v = t^{-2} \Delta^{P_1}_v, \Delta^{P_t}_h = \Delta^{P_1}_h$ and

$$\Delta^{P_t} = t^{-2} \Delta^{P_1} + (1 - t^{-2}) \Delta^{P_1}_h. \tag{1}$$

As a conclusion the eigenfunctions of Δ^{P_t} are eigenfunctions also for $\Delta^{P_1}_v, \Delta^{P_1}_h$ and Δ^{P_1}, and as such are independent on t. An eigenvalue of Δ^{P_t} can be written $\lambda(t) = \alpha + t^{-2}\beta$ where α is an eigenvalue for $\Delta^{P_1}_h$ and β is an eigenvalue for $\Delta^{P_1}_v$.

Now if ψ is an eigenfunction of Δ^{P_1} related to the eigenvalue $\alpha + \beta$ and if ϕ is an eigenfunction of $\Delta^{F_\infty} = \Delta^F$ related to $\mu \in \operatorname{Spec}(F_\infty) = \operatorname{Spec}(F)$ (see (ii)), then for the functions $\check{\psi} = \psi \circ pr_1, \check{\phi} = \phi \circ pr_2$ defined on $P \times F$ one has:

$$\Delta^{P_t \times F_t}(\check{\psi}\check{\phi}) = (\alpha + t^{-2}\beta + \mu)\check{\psi}\check{\phi} + \check{\psi}((\Delta^{F_t} - \Delta^{F_\infty})\check{\phi}). \tag{2}$$

It appears from (2) as t goes to infinity, that $\check{\psi}\check{\phi}$ *is an eigenfunction on* $P \times F$ *related to the eigenvalue* $\alpha + \mu$.

3. Eigenfunctions and eigenvalues of Δ^M

By $\mathcal{R}$ we denote the set of unitary irreducible (*complex*) representations of G. Since G is a compact Lie Group they are finite dimensional.

The group G acting on P and F isometrically, we have the decompositions

$$L^2(P) = \bigoplus_{\rho \in \mathcal{R}} L^2_\rho(P), \qquad L^2(F) = \bigoplus_{\rho \in \mathcal{R}} L^2_\rho(F), \tag{3}$$

where L^2_ρ are the isotypical components for the action of G in L^2 (they are orthogonal in L^2). We emphasize that the spaces under consideration are spaces of *complex valued* functions.

For a fixed representation ρ let $V \subset L^2_\rho(P)$ (resp. $W \subset L^2_\rho(F)$) be an irreducible (non zero) G-invariant space. Then by Schur's lemma there exists a unique, up to a complex number of modulus one, equivariant isometry between V and W; let us call it α_{VW}.

Now if $(\psi_1, \dots, \psi_d) = \Psi$ is an orthonormal basis of V ($d = \dim \rho$), then $\Phi = (\alpha_{VW}(\psi_i))$ is an orthonormal basis in W. We define the scalar product by

$$\langle \Psi, \Phi \rangle = \frac{1}{d} \sum_{i=1}^{d} \check{\psi}_i \check{\tilde{\phi}}_i, \qquad \text{where } \phi_i = \alpha_{VW}(\psi_i). \tag{4}$$

It is clear that $\langle \Psi, \Phi \rangle$, function on $P \times F$, is invariant under the action of G on $P \times F$ (see sect. 2). Furthermore it does not depend on the choice of Ψ.

For the sake of simplicity we can assume that we have chosen once for all an isomorphism α_{VW} and an orthonormal basis Ψ for each pair (V, W). We shall call such a pair (V, W), with the related isomorphism $\alpha_{V,W}$, *a canonical pair of the decomposition* (3), and moreover we shall say that (4) defines *the function on $P \times F$ asssociated to the canonical pair* (V, W).

Now we have the

Lemma. *The family of functions $\langle \Psi, \Phi \rangle$ on $P \times F$ defined by (4) forms an Hilbertian basis of $L^2_G(P \times F)$, the space of L^2-functions on $P \times F$ invariant under G.*

Proof. In fact we are just claiming that

$$L^2_G(P \times F) = \sum_{\rho \in \mathcal{R}} L^2_\rho(P) \otimes L^2_{\bar{\rho}}(F). \tag{5}$$

It is clear by computing L^2-scalar products that for different choices of (V, W) the corresponding functions $\langle \Psi, \Phi \rangle$ are orthogonal.

Now we have the decomposition

$$L^2(P \times F) = L^2(P) \otimes L^2(F) = \sum_{\rho, \rho' \in \mathcal{R}} L^2_\rho(P) \otimes L^2_{\rho'}(F)$$

then for $\eta \in L^2_G(P \times F)$ one can easily deduce from the classical orthogonality relations on the characters (see [9], p. 28, 48) that the orthogonal projection of η on $L^2_\rho(P) \otimes L^2_{\rho'}(F)$ is zero if $\rho' \neq \bar{\rho}$. ■

In the sequel we shall consider $\langle \Psi, \Phi \rangle$ as a function on M, quotient of $P \times F$ by the G action.

Since the structure group G acts on each space by isometries, the subspaces V and W (irreducible and G-invariant) are in fact included in some eigenspace. Let us assume now that V is included in an eigenspace E_V for Δ^{P_1} related to the eigenvalue $\alpha + \beta$,

$$\alpha \in \operatorname{Spec}(\Delta^{P_1}_h), \qquad \beta \in \operatorname{Spec}(\Delta^{P_1}_v) \tag{6}$$

and that W is in an eigenspace E_W for Δ^F related to μ,

$$\mu \in \operatorname{Spec}(\Delta^F). \tag{7}$$

Theorem. *The family of functions $\langle \Psi, \Phi \rangle$ associated to all the possible canonical pairs (V, W) is a Hilbertian basis of $L^2(M)$ consisting of eigenfunctions for Δ^M. Furthermore the eigenvalue corresponding to $\langle \Psi, \Phi \rangle$ is $\alpha + \mu$ (see (6, 7)).*

PROOF. The formula (2) applied to $\langle \Psi, \Phi \rangle$ instead of $\check{\psi}\check{\phi}$ gives that it is an eigenfunction if one let t go to infinity. Let us recall that $\Delta^{F_t} - \Delta^{F_\infty} = H_t$ the mean curvature of the fibers of ω, which goes to zero when t goes to infinity. ■

REMARKS

1) The theorem shows that not all sums $\alpha + \mu$ are eigenvalues of Δ^M. Indeed if α corresponds to a representation ρ, μ must correspond to $\bar{\rho}$. Now it is well known (see [5]) that *all* the unitary and irreducible representations of G appear in $L^2(P)$, but it is not the case for $L^2(F)$. It is well known (see [11]) that for a homogeneous space G/H, G a Lie group and H a closed subgroup (G is compact), one has the reciprocity formula $[L^2(G/H) : \rho] = [\rho_{|H} : 1]$ *i.e.*, the multiplicity of ρ in $L^2(G/H)$ is equal to the multiplicity of the trivial representation in the restriction of ρ to H.

Up to a set of measure zero, F is a G/H trivial fiber bundle over a manifold $\bar{F}$ (see [3]) thus $L^2(M) = L^2(\bar{F}) \times L^2(G/H)$ (see [5]). Here $\bar{F}$ is regarded as a trivial G-space.

A given representation appears in $L^2(F)$ if and only if it appears in $L^2(G/H)$. Here G/H is a maximal orbit type in F.

2) The multiplicity of a given eigenvalue ν of Δ^M is given by the formula

$$\text{mult}(\nu) = \sum_{\rho \in \mathcal{R}} \sum_{\alpha+\mu=\nu} m_\alpha(\rho) \cdot m_\mu(\bar{\rho}) \tag{8}$$

with $\alpha \in \text{Spec}(\Delta_h^P)$, $\mu \in \text{Spec}(\Delta^F)$ and where $m_\alpha(\rho)$ (resp. $m_\mu(\bar{\rho})$) is the multiplicity of the representation ρ (resp. $\bar{\rho}$) in the eigenspace of $\Delta_h^{P_1}$ (resp. Δ^F) related to the eigenvalue α (resp. μ). It is clearly finite.

3) Finally we see that the vertical Laplacian of P does not play any role. It is just a tool for computing the spectrum of Δ^M.

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